

On Classical Solutions of Linear Stochastic Integro-Differential Equations

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Abstract

We prove the existence of classical solutions to parabolic linear stochastic integro-differential equations with adapted coefficients using Feynman-Kac transformations, conditioning, and the interlacing of space-inverses of stochastic flows associated with the equations. The equations are forward and the derivation of existence does not use the “general theory” of SPDEs. Uniqueness is proved in the class of classical solutions with polynomial growth.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete filtered probability space and $\tilde{\mathcal{F}}_0$ be a sub-sigma-algebra of \mathcal{F} . We assume that this probability space supports a sequence $(w_t^{1;\varrho})_{\varrho \geq 1}$, $t \geq 0$, $\varrho \in \mathbf{N}$, of independent one-dimensional Wiener processes and a Poisson random measure $p^1(dt, dz)$ on $(\mathbf{R}_+ \times Z^1, \mathcal{B}(\mathbf{R}_+ \otimes Z^1))$ with intensity measure $\pi^1(dz)dt$, where $(Z^1, \mathcal{Z}^1, \pi^1)$ is a sigma-finite measure space. We also assume that $(w_t^{1;\varrho})_{\varrho \geq 1}$ and $p^1(dt, dz)$ are independent of $\tilde{\mathcal{F}}_0$. Let $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ be the standard augmentation of the filtration $(\tilde{\mathcal{F}}_t)_{t \geq 0}$, where for each $t \geq 0$,

$$\tilde{\mathcal{F}}_t = \sigma\left(\tilde{\mathcal{F}}_0, (w_s^1)_{\varrho \geq 1}, p^1([0, s], \Gamma) : s \leq t, \Gamma \in \mathcal{Z}^1\right).$$

For each real number $T > 0$, we let \mathcal{R}_T , \mathcal{O}_T , and \mathcal{P}_T be the \mathbf{F} -progressive, \mathbf{F} -optional, and \mathbf{F} -predictable sigma-algebra on $\Omega \times [0, T]$, respectively. Denote by $q^1(dt, dz) = p^1(dt, dz) - \pi^1(dz)dt$ the compensated Poisson random measure. Let $D^1, E^1, V^1 \in Z$ be disjoint \mathcal{Z}^1 -measurable subsets such that $D^1 \cup E^1 \cup V^1 = Z^1$ and $\pi(V^1) < \infty$. Let $(Z^2, \mathcal{Z}^2, \pi^2)$ be a sigma-finite measure space and $D^2, E^2 \in Z^2$ be disjoint \mathcal{Z}^2 -measurable subsets such that $D^2 \cup E^2 = Z^2$.

Fix an arbitrary positive real number $T > 0$ and integers $d_1, d_2 \geq 1$. Let $\alpha \in (0, 2]$ and let $\tau \leq T$ be a stopping time. Let \mathcal{F}_τ be the stopping time sigma-algebra associated with τ and let $\varphi : \Omega \times \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_2}$ be $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable. We consider the system of stochastic integro-differential equations on $[0, T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned} du_t^l &= \left((\mathcal{L}_t^{1;l} + \mathcal{L}_t^{2;l})u_t + \mathbf{1}_{[1,2]}(\alpha)b_i^l \partial_i u_t^l + c_i^{\bar{l}} u_t^{\bar{l}} + f_t^l \right) dt + \left(\mathcal{N}_t^{1;l\varrho} u_t + g_t^{l\varrho} \right) dw_t^{1;\varrho} \\ &\quad + \int_{Z^1} \left(\mathcal{I}_{t,z}^{1;l} u_{t-} + h_t^l(z) \right) [\mathbf{1}_{D^1}(z)q^1(dt, dz) + \mathbf{1}_{E^1 \cup V^1}(z)p^1(dt, dz)], \quad \tau \leq t \leq T, \\ u_t^l &= \varphi^l, \quad t \leq \tau, \quad l \in \{1, \dots, d_2\}, \end{aligned} \tag{1.1}$$

where for $\phi \in C_c^\infty(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$, $k \in \{1, 2\}$, and $l \in \{1, \dots, d_2\}$,

$$\begin{aligned} \mathcal{L}_t^{k;l} \phi(x) &:= \mathbf{1}_{\{2\}}(\alpha) \frac{1}{2} \sigma_t^{k;i\varrho}(x) \sigma_t^{k;j\varrho}(x) \partial_{ij} \phi^l(x) + \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{k;i\varrho}(x) v_t^{k;\bar{l}\varrho}(x) \partial_i \phi^{\bar{l}}(x) \\ &\quad + \int_{D^k} \rho_t^{k;\bar{l}\bar{l}}(x, z) \left(\phi^{\bar{l}}(x + H_t^k(x, z)) - \phi^{\bar{l}}(x) \right) \pi^k(dz) \\ &\quad + \int_{D^k} \left(\phi^l(x + H_t^k(x, z)) - \phi^l(x) - \mathbf{1}_{\{1,2\}}(\alpha) H_t^{k;i}(x, z) \partial_i \phi^l(x) \right) \pi^k(dz) \\ &\quad + \mathbf{1}_{\{2\}}(k) \int_{E^2} \left((I_{d_2}^{\bar{l}\bar{l}} + \rho_t^{2;\bar{l}\bar{l}}(x, z)) \phi^{\bar{l}}(x + H_t^2(x, z)) - \phi^{\bar{l}}(x) \right) \pi^2(dz), \\ \mathcal{N}_t^{1;l\varrho} \phi(x) &:= \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{1;i\varrho}(x) \partial_i \phi^l(x) + v_t^{1;\bar{l}\varrho}(x) \phi^{\bar{l}}(x), \quad \varrho \geq 1, \\ \mathcal{I}_{t,z}^{1;l} \phi(x) &:= (I_{d_2}^{\bar{l}\bar{l}} + \rho_t^{1;\bar{l}\bar{l}}(x, z)) \phi^{\bar{l}}(x + H_t^1(x, z)) - \phi^{\bar{l}}(x), \end{aligned}$$

and

$$\int_{D^k} \left(|H_t^k(x, z)|^\alpha + |\rho_t^k(x, z)|^2 \right) \pi^k(dz) + \int_{E^k} \left(|H_t^k(x, z)|^{1 \wedge \alpha} + |\rho_t^k(x, z)| \right) \pi^k(dz) < \infty.$$

The summation convention with respect to repeated indices $i, j \in \{1, \dots, d_1\}$, $\bar{l} \in \{1, \dots, d_2\}$, and $\varrho \in \mathbf{N}$ is used here and below. The $d_2 \times d_2$ dimensional identity matrix is denoted by I_{d_2} . For a subset A of a larger set X , $\mathbf{1}_A$ denotes the $\{0, 1\}$ -valued function taking the value 1 on the set A and 0 on the complement of A . We assume that for each $k \in \{1, 2\}$,

$$\sigma_t^k(x) = (\sigma_t^{k;i\varrho}(\omega, x))_{1 \leq i \leq d_1, \varrho \geq 1}, \quad b_t(x) = (b_t^i(\omega, x))_{1 \leq i \leq d_1}, \quad c_t(x) = (c_t^{l\bar{l}}(\omega, x))_{1 \leq l, \bar{l} \leq d_2},$$

$$v_t^k(x) = (v_t^{k;l\bar{l}\varrho}(\omega, x))_{1 \leq l, \bar{l} \leq d_2, \varrho \geq 1}, \quad f_t(x) = (f_t^i(\omega, x))_{1 \leq i \leq d_2}, \quad g_t(x) = (g_t^{i\varrho}(\omega, x))_{1 \leq i \leq d_2, \varrho \geq 1},$$

are random fields on $\Omega \times [0, T] \times \mathbf{R}^{d_1}$ that are $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable. Moreover, for each $k \in \{1, 2\}$, we assume that

$$H_t^k(x, z) = (H_t^{k;i}(\omega, x, z))_{1 \leq i \leq d_1}, \quad \rho_t^k(x, z) = (\rho_t^{k;l\bar{l}}(\omega, x, z))_{1 \leq l, \bar{l} \leq d_2},$$

are random fields on $\Omega \times [0, T] \times \mathbf{R}^{d_1} \times \mathbf{Z}^k$ that are $\mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}^k$ -measurable. Moreover, we assume that

$$h_t(x, z) = (h_t^i(\omega, x, z))_{1 \leq i \leq d_2},$$

is a random field on $\Omega \times [0, T] \times \mathbf{R}^{d_1}$ that is $\mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable

Systems of linear stochastic integro-differential equations appear in many contexts. They may be considered as extensions of both first-order symmetric hyperbolic systems and linear fractional advection-diffusion equations. The equation (1.1) also arises in non-linear filtering of semimartingales as the equation for the unnormalized filter of the signal (see, e.g., [Gri76] and [GM11]). Moreover, (1.1) is intimately related to linear transformations of inverse flows of jump SDEs and it is precisely this connection that we will exploit to obtain solutions.

There are various techniques available to derive the existence and uniqueness of classical solutions of linear parabolic SPDEs and SDEs. One approach is to develop a theory of weak solutions for the equations (e.g. variational, mild solution, or etc...) and then study further regularity in classical function spaces via an embedding theorem. We refer the reader to [Par72, Par75, MP76, KR77, Tin77, Gyö82, Wal86, DPZ92, Kry99, CK10, PZ07, Hau05, RZ07, BvNVW08, HØUZ10, LM14a] for more information about weak solutions of SPDEs driven by continuous and discontinuous martingales and martingale measures. This approach is especially important in the non-degenerate setting where some smoothing occurs and has the obvious advantage that it is broader in scope. Another approach is to regard the solution as a function with values in a probability space and use the method deterministic PDEs (i.e. Schauder estimates, see, e.g. [Mik00, MP09]). A third approach is a direct one that uses solutions of stochastic differential equations. The direct method allows to obtain classical solutions in the entire Hölder scale while not restricting to integer derivative assumptions for the coefficients and data.

In this paper, we derive the existence of a classical solutions of (1.1) with regular coefficients using a Feynman-Kac-type transformation and the interlacing of the space-inverse (first integrals [KR81]) of a stochastic flow associated with the equation. The construction of the solution gives an insight into the structure of the solution as well. We prove that the solution of (1.1) is unique in the class of classical solutions with polynomial growth

(i.e. weighted Hölder spaces). As an immediate corollary of our main result, we obtain the existence and uniqueness of classical solutions of linear integro-differential equations with random coefficients, since the coefficients σ^1 , H^1 , a^1 , ρ^1 , and free terms g and h can be zero. Our work here directly extends the method of characteristics for deterministic first-order partial differential equations and the well-known Feynman-Kac formula for deterministic second-order PDEs.

In the continuous case (i.e. $H^1 \equiv 0, H^2 \equiv 0, h \equiv 0$), the classical solution of (1.1) was constructed in [KR81, Kun81, Kun86, Roz90] (see references therein as well) using the first integrals of the associated backward SDE. This method was also used to obtain classical solutions of (1.1) in [DPMT07]. In the references above, the forward Liouville equation for the first integrals of associated stochastic flow was derived directly. However, since the backward equation involves a time reversal, the coefficients and input functions are assumed to be non-random. The generalized solutions of (1.1) with $d_2 = 1$, non-random coefficients, non-degenerate diffusion, and finite measures $\pi^1 = \pi^2$ were discussed in [MB07]. In this paper, we give a direct derivation of (1.1) and all the equations considered are forward, possibly degenerate, and the coefficients and input functions are adapted. For other interesting and related developments, we refer the reader to [Pri12, Zha13, Pri14].

This paper is organized as follows. In Section 2, our notation is set forth and the main results are stated. In Section 3, the main theorem is proved and is divided into a proof of uniqueness and existence. In Section 4, the appendix, auxiliary facts that are used throughout the paper are discussed.

2 Outline of main results

For each integer $n \geq 1$, let \mathbf{R}^n be the space of d -dimensional Euclidean points $x = (x^1, \dots, x^n)$. For each x , denote by $|x|$ the Euclidean norm of x . Let \mathbf{R}_+ denote the set of non-negative real-numbers. Let \mathbf{N} be the set of natural numbers. Elements of \mathbf{R}^{d_1} and \mathbf{R}^{d_2} are understood as column vectors and elements of \mathbf{R}^{2d_1} and \mathbf{R}^{2d_2} are understood as matrices of dimension $d_1 \times d_1$ and $d_2 \times d_2$, respectively. For each integer $n \geq 1$, the norm of an element x of $\ell_2(\mathbf{R}^n)$, the space of square-summable \mathbf{R}^n -valued sequences, is denoted by $|x|$. For a topological space (X, \mathcal{X}) we denote the Borel sigma-field on X by $\mathcal{B}(X)$.

For each $i \in \{1, \dots, d_1\}$, let $\partial_i = \frac{\partial}{\partial x_i}$ be the spatial derivative operator with respect to x_i and write $\partial_{ij} = \partial_i \partial_j$ for each $i, j \in \{1, \dots, d_1\}$. For a once differentiable function $f = (f^1, \dots, f^{d_1}) : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_1}$, we denote the gradient of f by $\nabla f = (\partial_j f^i)_{1 \leq i, j \leq d_1}$. Similarly, for a once differentiable function $f = (f^{1\varrho}, \dots, f^{d\varrho})_{\varrho \geq 1} : \mathbf{R}^{d_1} \rightarrow \ell_2(\mathbf{R}^{d_1})$, we denote the gradient of f by $\nabla f = (\partial_j f^{i\varrho})_{1 \leq i, j \leq d_1, \varrho \geq 1}$ and understand it as a function from \mathbf{R}^{d_1} to $\ell_2(\mathbf{R}^{2d_1})$. For a multi-index $\gamma = (\gamma_1, \dots, \gamma_d) \in \{0, 1, 2, \dots\}^d$ of length $|\gamma| := \gamma_1 + \dots + \gamma_d$, denote by ∂^γ the operator $\partial^\gamma = \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d}$, where ∂_i^0 is the identity operator for all $i \in \{1, \dots, d_1\}$. For each integer $d \geq 1$, we denote by $C_c^\infty(\mathbf{R}^{d_1}; \mathbf{R}^d)$ the space of infinitely differentiable functions with compact support in \mathbf{R}^d .

For a Banach space V with norm $|\cdot|_V$, domain Q of \mathbf{R}^d , and continuous function $f : Q \rightarrow$

V , we define

$$\|f\|_{0;Q;V} = \sup_{x \in Q} |f(x)|$$

and

$$[f]_{\beta;Q;V} = \sup_{x,y \in Q, x \neq y} \frac{|f(x) - f(y)|_V}{|x - y|_V^\beta}, \quad \beta \in (0, 1].$$

For each real number $\beta \in \mathbf{R}$, we write $\beta = [\beta]^- + \{\beta\}^+$, and $\{\beta\}^+ \in (0, 1]$. For a Banach space V with norm $|\cdot|_V$, real number $\beta > 0$, and domain Q of \mathbf{R}^d , we denote by $C^\beta(Q; V)$ the Banach space of all bounded continuous functions $f : Q \rightarrow V$ having finite norm

$$\|f\|_{\beta;Q;V} := \sum_{|\gamma| \leq [\beta]^-} |\partial^\gamma f|_{0;Q;V} + \sum_{|\gamma| = [\beta]^-} [\partial^\gamma f]_{\{\beta\}^+;Q;V}.$$

When $Q = \mathbf{R}^{d_1}$ and $V = \mathbf{R}^n$ or $V = \ell_2(\mathbf{R}^n)$ for any integer $n \geq 1$, we drop the subscripts Q and V from the norm $|\cdot|_{\beta;Q;V}$ and write $|\cdot|_\beta$. For a Banach space V and for each $\beta > 0$, denote by $C_{loc}^\beta(\mathbf{R}^d; V)$ the Fréchet space of continuous functions $f : \mathbf{R}^d \rightarrow V$ satisfying $f \in C^\beta(Q; V)$ for all bounded domains $Q \subset \mathbf{R}^d$. We call a function $f : \mathbf{R}^d \rightarrow \mathbf{R}^d$ a $C_{loc}^\beta(\mathbf{R}^d; \mathbf{R}^d)$ -diffeomorphism if f is a homeomorphism and both f and its inverse f^{-1} are in $C_{loc}^\beta(\mathbf{R}^d; \mathbf{R}^d)$.

For a Fréchet space χ , we denote by $D([0, T]; \chi)$ the space of χ -valued càdlàg functions on $[0, T]$. Unless otherwise specified, we endow $D([0, T]; \chi)$ with the supremum seminorms.

The notation $N = N(\cdot, \dots, \cdot)$ is used to denote a positive constant depending only on the quantities appearing in the parentheses. In a given context, the same letter is often used to denote different constants depending on the same parameter. If we do not specify to which space the parameters ω, t, x, y, z and n belong, then we mean $\omega \in \Omega$, $t \in [0, T]$, $x, y \in \mathbf{R}^{d_1}$, $z \in Z^k$, and $n \in \mathbf{N}$.

Let $r_1(x) := \sqrt{1 + |x|^2}$, $x \in \mathbf{R}^{d_1}$. Let us introduce some regularity conditions on the coefficients and free terms. We consider these assumptions for $\bar{\beta} > 1 \vee \alpha$ and $\bar{\beta} > \alpha$.

Assumption 2.1 ($\bar{\beta}$). (1) *There is a constant $N_0 > 0$ such that for each $k \in \{1, 2\}$ and all $\omega, t \in \Omega \times [0, T]$,*

$$|r_1^{-1} b_t|_0 + |\nabla b_t|_{\bar{\beta}-1} + |r_1^{-1} \sigma_t^k|_0 + |\nabla \sigma_t^k|_{\bar{\beta}-1} \leq N_0.$$

Moreover, for each $k \in \{1, 2\}$ and all $(\omega, t, z) \in \Omega \times [0, T] \times (D^k \cup E^k)$,

$$|r_1^{-1} H_t^k(z)|_0 \leq K_t^k(z) \quad \text{and} \quad |\nabla H_t^k(z)|_{\bar{\beta}-1} \leq \bar{K}_t^k(z)$$

where $K^k, \bar{K}^k : \Omega \times [0, T] \times (D^k \cup E^k) \rightarrow \mathbf{R}_+$ are $\mathcal{P}_T \otimes \mathcal{Z}^k$ -measurable functions satisfying

$$K_t^k(z) + \bar{K}_t^k(z) + \int_{D^k} (K_t^k(z)^\alpha + \bar{K}_t^k(z)^2) \pi^k(dz) + \int_{E^k} (K_t^k(z)^{1 \wedge \alpha} + \bar{K}_t^k(z)) \pi^k(dz) \leq N_0,$$

for all $(\omega, t, z) \in \Omega \times [0, T] \times (D^k \cup E^k)$.

(2) For each $k \in \{1, 2\}$, there is a constant $\eta^k \in (0, 1)$ such that for all $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^{d_1} \times (D^k \cup E^k) : |\nabla H_t^k(\omega, x, z)| > \eta^k\}$,

$$\left| \left(I_{d_1} + \nabla H_t^k(x, z) \right)^{-1} \right| \leq N_0.$$

Assumption 2.2 ($\tilde{\beta}$). There is a constant $N_0 > 0$ such that for each $k \in \{1, 2\}$ and all $(\omega, t) \in \Omega \times [0, T]$,

$$|c_t|_{\tilde{\beta}} + |v_t^k|_{\tilde{\beta}} + |r_1^{-\theta} f_t|_{\tilde{\beta}} + |r_1^{-\theta} g_t|_{\tilde{\beta}} \leq N_0.$$

Moreover, for each $k \in \{1, 2\}$ and all $(\omega, t, z) \in \Omega \times [0, T] \times (D^k \cup E^k)$,

$$|\rho_t^k(z)|_{\tilde{\beta}} \leq l_t^k(z), \quad |r_1^{-\theta} h_t(z)|_{\tilde{\beta}} \leq l_t^k(z),$$

where $l^k : \Omega \times [0, T] \times Z^k \rightarrow \mathbf{R}_+$ are $\mathcal{P}_T \otimes \mathcal{Z}^k$ -measurable function satisfying

$$l_t^k(z) + \int_{D^k} l_t^k(z)^2 \pi^k(dz) + \int_{E^k} l_t^k(z) \pi^k(dz) \leq N_0,$$

for all $(\omega, t, z) \in \Omega \times [0, T] \times (D^k \cup E^k)$.

Remark 2.1. It follows from Lemma 4.10 and Remark 4.11 that if Assumption 2.1($\bar{\beta}$) holds for some $\bar{\beta} > 1 \vee \alpha$, then for all ω, t , and $z \in D^k \cup E^k$, $x \mapsto \tilde{H}_t^k(x, z) := x + H_t^k(x, z)$ is a diffeomorphism.

Let Assumptions 2.1($\bar{\beta}$) and 2.2($\tilde{\beta}$) hold for some $\bar{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$. In our derivation of a solutions of (1.1), we first obtain solutions of equations of a special form. Specifically, consider the system of SDEs on $[0, T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned} d\hat{u}_t^l &= \left((\mathcal{L}_t^{1;l} + \mathcal{L}_t^{2;l}) \hat{u}_t + \hat{b}_t^l \partial_i u_t^l + \hat{c}_t^{\bar{l}} u_t^{\bar{l}} + \hat{f}_t^l \right) dt + \left(\mathcal{N}_t^{1;l\varrho} \hat{u}_t + g_t^{l\varrho} \right) dw_t^{1;\varrho} \\ &\quad + \int_{Z^1} \left(I_{t,z}^{1;l} \hat{u}_{t-} + h_t^l(z) \right) [\mathbf{1}_{D^1}(z) q^1(dt, dz) + \mathbf{1}_{E^1}(z) p^1(dt, dz)], \quad \tau < t \leq T, \\ \hat{u}_t^l &= \varphi_t^l, \quad t \leq \tau, \quad l \in \{1, \dots, d_2\}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \hat{b}_t^l(x) &:= \mathbf{1}_{[1,2]}(\alpha) b_t^l(x) + \sum_{k=1}^2 \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{k;j\varrho}(x) \partial_j \sigma_t^{k;i\varrho}(x) \\ &\quad + \sum_{k=1}^2 \mathbf{1}_{(1,2]}(\alpha) \int_{D^k} \left(H_t^{k;i}(x, z) - H_t^{k;i}(\tilde{H}_t^{k;-1}(x, z), z) \right) \pi^k(dz), \\ \hat{c}_t^{\bar{l}}(x) &:= c_t^{\bar{l}}(x) + \sum_{k=1}^2 \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{k;j\varrho}(x) \partial_j v_t^{k;\bar{l}\varrho}(x) \\ &\quad + \sum_{k=1}^2 \int_{D^k} \left(\rho_t^{k;\bar{l}}(x, z) - \rho_t^{k;\bar{l}}(\tilde{H}_t^{k;-1}(x, z), z) \right) \pi^k(dz), \\ \hat{f}_t^l(x) &:= f_t^l(x) + \sigma_t^{1;j\varrho}(x) \partial_j g_t^{l\varrho}(x) + \int_{D^1} \left(h_t^l(x, z) - h_t^l(\tilde{H}_t^{1;-1}(x, z), z) \right) \pi^1(dz). \end{aligned}$$

Let $(w_t^{2;\varrho})_{\varrho \geq 1}$, $t \geq 0$, $\varrho \in \mathbf{N}$, be a sequence of independent one-dimensional Wiener processes. Let $p^2(dt, dz)$ be a Poisson random measure on $([0, \infty) \times \mathcal{Z}^2, \mathcal{B}([0, \infty) \otimes \mathcal{Z}^2))$ with intensity measure $\pi^2(dz)dt$. Extending the probability space if necessary, we take w^2 and $p^2(dt, dz)$ to be independent of w^1 and $p^1(dt, dz)$. Let

$$\hat{\mathcal{F}}_t = \sigma\left((w_s^2)_{\varrho \geq 1}, p^2([0, s], \Gamma) : s \leq t, \Gamma \in \mathcal{Z}^2\right)$$

and $\tilde{\mathbf{F}} = (\tilde{\mathcal{F}}_t)_{t \leq T}$ be the standard augmentation of $(\mathcal{F}_t \vee \hat{\mathcal{F}}_t)_{t \leq T}$. Denote by $q^2(dt, dz) = p^2(dt, dz) - \pi^2(dz)dt$ the compensated Poisson random measure. We associate with the SIDE (2.1), the $\tilde{\mathbf{F}}$ -adapted stochastic flow $X_t = X_t(x) = X_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, generated by the SDE

$$\begin{aligned} dX_t &= -\mathbf{1}_{[1,2]}(\alpha)b_t(X_t)dt + \sum_{k=1}^2 \mathbf{1}_{[2]}(\alpha)\sigma_t^{k;\varrho}(X_t)dw_t^{k;\varrho} \\ &\quad - \sum_{k=1}^2 \int_{D^k} H_t^k(\tilde{H}_t^{k;-1}(X_{t-}, z), z)[p^k(dt, dz) - \mathbf{1}_{(1,2]}(\alpha)\pi^k(dz)dt] \\ &\quad - \sum_{k=1}^2 \int_{E^k} H_t^k(\tilde{H}_t^{k;-1}(X_{t-}, z), z)p^k(dt, dz), \quad \tau < t \leq T, \\ X_t &= x, \quad t \leq \tau, \end{aligned} \tag{2.2}$$

and the $\tilde{\mathbf{F}}$ -adapted random field $\Phi_t(x) = \Phi_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, solving the linear SDE given by

$$\begin{aligned} d\Phi_t(x) &= (c_t(X_t(x))\Phi_t(x) + f_t(X_t(x)))dt + \sum_{k=1}^2 v_t^{k;\varrho}(X_t(x))\Phi_t(x)dw_t^{k;\varrho} + g_t^{\varrho}(X_t(x))dw_t^{1;\varrho} \\ &\quad + \sum_{k=1}^2 \int_{Z^k} \rho_t^k(\tilde{H}_t^{k;-1}(X_{t-}(x), z), z)\Phi_{t-}(x)[\mathbf{1}_{D^k}(z)q^k(dt, dz) + \mathbf{1}_{E^k}(z)p^k(dt, dz)] \\ &\quad + \int_{Z^1} h_t(\tilde{H}_t^{1;-1}(X_{t-}(x), z), z)[\mathbf{1}_{D^1}(z)q^1(dt, dz) + \mathbf{1}_{E^1}(z)p^1(dt, dz)], \quad \tau < t \leq T, \\ \Phi_t(x) &= \varphi(x), \quad t \leq \tau. \end{aligned}$$

The coming theorem is our existence, uniqueness, and representation theorem for (2.1). Let us describe our solution class. For each $\beta' \in (0, \infty)$, denote by $\mathfrak{U}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ the linear space of all \mathbf{F} -adapted random fields $v = v_t(x)$ such that \mathbf{P} -a.s.

$$\mathbf{1}_{[\tau_n, \tau_{n+1})} r_1^{-\lambda_n} v \in D([0, T]; \mathcal{C}^{\beta'}(\mathbf{R}^{d_1}, \mathbf{R}^{d_2})),$$

where $(\tau_n)_{n \geq 0}$ is an increasing sequence of \mathbf{F} -stopping times with $\tau_0 = 0$ and $\tau_n = T$ for sufficiently large n , and where for each n , λ_n is a positive \mathcal{F}_{τ_n} -measurable random variable.

Theorem 2.2. *Let Assumptions 2.1($\bar{\beta}$) and 2.2($\tilde{\beta}$) hold for some $\bar{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$. For each stopping time $\tau \leq T$ and $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable random field φ such that for some $\beta' \in (\alpha, \bar{\beta} \wedge \tilde{\beta})$ and $\theta' \geq 0$, \mathbf{P} -a.s. $r_1^{-\theta'} \varphi \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$, there exists a unique solution $\hat{u} = \hat{u}(\tau)$ of (2.1) in $\mathfrak{U}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ and for all $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, \mathbf{P} -a.s.*

$$\hat{u}_t(\tau, x) = \mathbf{E} \left[\Phi_t(\tau, X_t^{-1}(\tau, x)) | \mathcal{F}_t \right]. \quad (2.3)$$

Moreover, for each $\epsilon > 0$ and $p \geq 2$,

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-\theta \vee \theta' - \epsilon} \hat{u}_t(\tau)|^p | \mathcal{F}_\tau \right] \leq N(|r_1^{-\theta'} \varphi|_{\beta'}^p + 1), \quad (2.4)$$

for a constant $N = N(d_1, d_2, p, N_0, T, \beta', \eta^1, \eta^2, \epsilon, \theta, \theta')$.

Using Itô's formula it is easy to check that if $m = 1$ and

$$g_t(x) = 0, \quad h_t(x) = 0, \quad \text{and} \quad \rho_t^k(x, z) \geq -1,$$

for all $(\omega, t, x, z) \in \Omega \times [[\tau, T]] \times \mathbf{R}^{d_1} \times (D^k \cup E^k)$, $k \in \{1, 2\}$, then

$$\Phi_t(x) = \Psi_t(x)\phi(x) + \Psi_t(x) \int_{[\tau, \tau \vee t]} \Psi_s^{-1}(x) f_s(X_s(x)) ds,$$

where \mathbf{P} -a.s. for all t and x ,

$$\begin{aligned} \Psi_t(x) = & e^{\int_{[\tau, \tau \vee t]} (c_s(X_s(x)) - \sum_{k=1}^2 \frac{1}{2} v_s^{k, \varrho}(X_s(x)) v_s^{k, \varrho}(X_s(x)) ds + \sum_{k=1}^2 \int_{[\tau, \tau \vee t]} v_s^{k, \varrho}(X_s(x)) dw_s^{k, \varrho} \\ & \cdot e^{-\sum_{k=1}^2 \int_{[\tau, \tau \vee t]} \int_{D^k} (\ln(1 + \rho_s^k(\tilde{H}_s^{k, -1}(X_{s-}(x), z), z)) - \rho_s^k(\tilde{H}_s^{k, -1}(X_{s-}(x), z), z)) \pi^k(dz) ds \\ & \cdot e^{\sum_{k=1}^2 \int_{[\tau, \tau \vee t]} \int_{Z^k} \ln(1 + \rho_s^k(\tilde{H}_s^{k, -1}(X_{s-}(x), z), z)) [\mathbf{1}_{D^k}(z) q^k(ds, dz) + \mathbf{1}_{E^k}(z) p^k(ds, dz)]}. \end{aligned} \quad (2.5)$$

The following corollary then follows directly from (2.3) and the (2.5).

Corollary 2.3. *Let $m = 1$ and assume that*

$$g_t(x) = 0, \quad h_t(x, z) = 0, \quad \rho_t^k(x, z) \geq -1, \quad \forall (\omega, t, x, z) \in [[\tau, T]] \times \mathbf{R}^{d_1} \times (D^k \cup E^k), \quad k \in \{1, 2\}.$$

Moreover, let Assumptions 2.1($\bar{\beta}$) and 2.2($\tilde{\beta}$) hold for some $\bar{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$. Let $\tau \leq T$ be stopping time and φ be a $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable random field such that for some $\beta' \in (\alpha, \bar{\beta} \wedge \tilde{\beta})$ and $\theta' \geq 0$, \mathbf{P} -a.s. $r_1^{-\theta'} \varphi \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$.

- (1) *If for all $(\omega, t, x) \in [[\tau, T]] \times \mathbf{R}^{d_1}$, $f_t(x) \geq 0$ and $\varphi(x) \geq 0$, then the solution \hat{u} of (1.1) satisfies $\hat{u}_t(x) \geq 0$, \mathbf{P} -a.s. for all $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$.*
- (2) *If for all $(\omega, t, x, z) \in [[\tau, T]] \times \mathbf{R}^{d_1} \times (D^k \cup E^k)$, $k \in \{1, 2\}$, $v_t^k(x) = 0$, $f_t(x) \leq 0$, $c_t(x) \leq 0$, $\varphi(x) \leq 1$, and $\rho_t^k(x, z) \leq 0$, then the solution \hat{u} of (1.1) satisfies $\hat{u}_t(x) \leq 1$, \mathbf{P} -a.s. for all $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$.*

Remark 2.4. Since \mathcal{L}^2 can be the zero operator, both Theorem 2.2 and Corollary 2.3 apply to fully degenerate equations and partial differential equations with random coefficients.

Now, let us discuss our existence and uniqueness theorem for (1.1). We construct the solution of $u = u(\tau)$ of (1.1) by interlacing the solutions of (2.1) along a sequence of large jump moments (see Section 3.5). By using an interlacing procedure we are also able to drop the condition of boundedness of $(I + \nabla H_t^1(x, z))^{-1}$ on the set $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^{d_1} \times (D^1 \cup E^1) : |\nabla H_t^1(\omega, x, z)| > \eta^k\}$. Also, in order to remove the terms in \hat{b} , \hat{c} , and \hat{f} that appear in (2.1), but not in (1.1), we subtract terms from the relevant coefficients in the flow and the transformation. However, in order to do this, we need to impose stronger regularity assumptions on some of the coefficients and free terms. We will introduce the parameters $\mu^1, \mu^2, \delta^1, \delta^2 \in [0, \frac{\alpha}{2}]$, which essentially allows one to trade-off integrability in z and regularity in x of the coefficients $H_t^k(x, z), \rho_t^k(x, z), h_t^k(x, z)$. It is worth mentioning that the removal of terms and the interlacing procedure are independent of each other and that it is due only to the weak assumptions on H^1 and ρ^1 on the set V^1 that we do not have moment estimates and a simple representation property like (2.4) for the solution of (1.1). Nevertheless, there is a representation of sorts and we refer the reader to the proof of the coming theorem for an explicit construction of the solution.

We introduce the following assumption for $\bar{\beta} > 1 \vee \alpha$, $\tilde{\beta} > \alpha$, and $\delta^1, \delta^2, \mu^1, \mu^2 \in [0, \frac{\alpha}{2}]$.

Assumption 2.3 ($\bar{\beta}, \mu^1, \mu^2, \delta^1, \delta^2$). (1) *There is a constant $N_0 > 0$ such that for each $k \in \{1, 2\}$ and all $(\omega, t) \in \Omega \times [0, T]$,*

$$|r_1^{-1}b_t|_0 + |\nabla b_t|_{\bar{\beta}-1} + |\sigma_t^k|_{\bar{\beta}+1} \leq N_0.$$

(2) *For each $k \in \{1, 2\}$ and all $(\omega, t) \in \Omega \times [0, T]$,*

$$\begin{aligned} |H_t^k(z)|_0 &\leq K_t^k(z), \quad |\nabla H_t^k(z)|_{\bar{\beta}-1}, \quad \forall z \in D^k, \\ |r_1^{-1}H_t^k(z)|_0 &\leq K_t^k(z), \quad |\nabla H_t^k(z)|_{\bar{\beta}-1} \leq \bar{K}_t^k(z), \quad \forall z \in E^k, \\ |\rho^k(t, z)|_{\bar{\beta}} &\leq l_t^k(z), \quad \forall z \in D^k, \quad |r_1^{-\theta}h_t(z)|_{\bar{\beta}} \leq l_t^1(z), \quad \forall z \in D^1, \end{aligned}$$

where $K^k, \bar{K}^k, l^k : \Omega \times [0, T] \times (D^k \cup E^k) \rightarrow \mathbf{R}_+$ are $\mathcal{P}_T \otimes \mathcal{Z}^k$ -measurable functions satisfying for all $(\omega, t, z) \in \Omega \times [0, T] \times (D^k \cup E^k)$,

$$K_t^k(z) + \bar{K}_t^k(z) + l_t^k(z) \leq N_0$$

and

$$\int_{D^k} (K_t^k(z)^\alpha + \bar{K}_t^k(z)^2 + l_t^k(z)^2) \pi^k(dz) + \int_{E^k} (K_t^k(z)^{1 \wedge \alpha} + \bar{K}_t^k(z)) \pi^k(dz) \leq N_0.$$

(3) For each $k \in \{1, 2\}$ and all $(\omega, t) \in \Omega \times [0, T]$,

$$\begin{aligned}
& |v_t^k|_{\bar{\beta}+1} \leq N_0, \text{ if } \sigma_t^k \neq 0, \quad |g_t|_{\bar{\beta}+1} \leq N_0, \text{ if } \sigma_t^1 \neq 0, \\
& \sum_{|\gamma|=[\bar{\beta}]^-} |\partial^\gamma H_t^k(z)|_{\{\bar{\beta}\}^+ + \delta^k} \leq \tilde{K}_t^k(z), \quad \forall z \in D^k, \text{ if } \{\bar{\beta}\}^+ + \delta^k \leq 1, \\
& \sum_{|\gamma|=[\bar{\beta}]^-} |\nabla \partial^\gamma H_t^k(z)|_0 \leq \bar{K}_t^k(z), \quad \sum_{|\gamma|=[\bar{\beta}]^-} |\nabla \partial^\gamma H_t^k(z)|_{\{\bar{\beta}\}^+ + \delta^k - 1} \leq \tilde{K}_t^k(z), \quad \forall z \in D^k, \text{ if } \{\bar{\beta}\}^+ + \delta^k > 1, \\
& \sum_{|\gamma|=[\bar{\beta}]^-} |\partial^\gamma \rho_t^k(z)|_{\{\bar{\beta}\}^+ + \mu^k} \leq \tilde{l}_t^k(z), \quad \forall z \in D^k, \text{ if } \{\bar{\beta}\}^+ + \mu^k \leq 1, \\
& \sum_{|\gamma|=[\bar{\beta}]^-} |\nabla \partial^\gamma \rho_t^k(z)|_0 \leq l_t^k(z), \quad \sum_{|\gamma|=[\bar{\beta}]^-} |\nabla \partial^\gamma \rho_t^k(z)|_{\{\bar{\beta}\}^+ + \mu^k - 1} \leq \tilde{l}_t^k(z), \quad \forall z \in D^k, \text{ if } \{\bar{\beta}\}^+ + \mu^k > 1, \\
& \sum_{|\gamma|=[\bar{\beta}]^-} |\partial^\gamma h_t^1(z)|_{\{\bar{\beta}\}^+ + \mu^1} \leq \tilde{l}_t^1(z), \quad \forall z \in D^1, \text{ if } \{\bar{\beta}\}^+ + \mu^1 \leq 1, \\
& \sum_{|\gamma|=[\bar{\beta}]^-} |\nabla \partial^\gamma h_t^1(z)|_0 \leq l_t^1(z), \quad \sum_{|\gamma|=[\bar{\beta}]^-} |\nabla \partial^\gamma h_t^1(z)|_{\{\bar{\beta}\}^+ + \mu^1 - 1} \leq \tilde{l}_t^1(z), \quad \forall z \in D^1, \text{ if } \{\bar{\beta}\}^+ + \mu^1 > 1,
\end{aligned}$$

where $\tilde{K}^k, \tilde{l}^k : \Omega \times [0, T] \times D^k \rightarrow \mathbf{R}_+$ are $\mathcal{P}_T \otimes \mathcal{Z}^k$ -measurable functions satisfying for all $(\omega, t, z) \in \Omega \times [0, T] \times D^k$,

$$\tilde{K}_t^k(z) + \tilde{l}_t^k(z) + \int_{D^k} \left(\tilde{K}_t^k(z)^{\frac{\alpha}{\alpha-\delta^k}} \mathbf{1}_{[0, \frac{\alpha}{2}]}(\delta^k) + \tilde{K}_t^k(z)^2 + \tilde{l}_t^k(z)^{\frac{\alpha}{\alpha-\mu^k}} \mathbf{1}_{[0, \frac{\alpha}{2}]}(\mu^k) + \tilde{l}_t^k(z)^2 \right) \pi^k(dz) \leq N_0.$$

(4) There is a constant $\eta^2 \in (0, 1)$ such that for all $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z^2 : |\nabla H_t^2(\omega, x, z)| > \eta^2\}$,

$$\left| \left(I_{d_1} + \nabla H_t^2(x, z) \right)^{-1} \right| \leq N_0.$$

Assumption 2.4 ($\tilde{\beta}$). (1) There is a constant $N_0 > 0$ such that for each $k \in \{1, 2\}$ and all $(\omega, t) \in \Omega \times [0, T]$,

$$\begin{aligned}
& |c_t|_{\tilde{\beta}} + |r_1^{-\theta} f_t|_{\tilde{\beta}} \leq N_0, \\
& |v_t^k|_{\tilde{\beta}} \leq N_0, \text{ if } \sigma_t^k = 0, \quad |g_t|_{\tilde{\beta}} \leq N_0, \text{ if } \sigma_t^1 = 0, \\
& |\rho^k(t, z)|_{\tilde{\beta}} \leq l_t^k(z), \quad \forall z \in E^k, \quad |r_1^{-\theta} h_t(z)|_{\tilde{\beta}} \leq l_t^1(z), \quad \forall z \in E^1,
\end{aligned}$$

where for all $(\omega, t) \in \Omega \times [0, T]$, $\int_{E^k} l_t^k(z) \pi^k(dz) \leq N_0$.

(2) There exist processes $\xi, \zeta : \Omega \times [0, T] \times V^1 \rightarrow \mathbf{R}_+$ that are $\mathcal{P}_T \otimes \mathcal{Z}^1$ -measurable satisfying

$$|r_1^{-\xi_t(z)} H_t^1(z)|_{\tilde{\beta} \vee 1} + |r_1^{-\xi_t(z)} \rho_t^1(z)|_{\tilde{\beta}} + |r_1^{-\xi_t(z)} h_t(z)|_{\tilde{\beta}} \leq \zeta_t(z),$$

for all $(\omega, t, z) \in \Omega \times [0, T] \times V^1$.

We now state our existence and uniqueness theorem for (1.1).

Theorem 2.5. Let Assumptions 2.3($\tilde{\beta}, \delta^1, \delta^2, \mu^1, \mu^2$) and 2.4($\tilde{\beta}$) hold for some $\tilde{\beta} > 1 \vee \alpha$, $\tilde{\beta} > \alpha$, and $\delta^1, \delta^2, \mu^1, \mu^2 \in [0, \frac{\alpha}{2}]$. For each stopping time $\tau \leq T$ and $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable random field φ such that for some $\beta' \in (\alpha, \tilde{\beta} \wedge \tilde{\beta})$ and $\theta' \geq 0$, \mathbf{P} -a.s. $r_1^{-\theta'} \varphi \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$, there exists a unique solution $u = u(\tau)$ of (1.1) in $\mathcal{C}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$.

3 Proof of main theorems

We will first prove uniqueness of the solution of (2.1) in the class $\mathfrak{U}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$. The existence part of the proof of Theorem 2.2 is divided into a series of steps. In the first step, by appealing to the representation theorem we derived for solutions of continuous SPDEs in Theorem 2.4 in [LM14b], we use an interlacing procedure and the strong limit theorem given in Theorem 2.3 in [LM14b] to show that the space inverse of the flow generated by a jump SDE (i.e. the SDE (2.2) without the uncorrelated noise) solves a degenerate linear SIDE. Then we linearly transform the inverse flow of a jump SDE to obtain solutions of degenerate linear SIDEs with free and zero-order terms and an initial condition. In the last step of the proof of Theorem 2.2, we introduce an independent Wiener process and Poisson random measure as explained above, apply the results we know for fully degenerate equations, and then take the optional projection of the equation. In the last section, Section 3.4, we prove Theorem 2.5 using an interlacing procedure and removing the extra terms in \hat{b} , \hat{c} and \hat{f} . The uniqueness of the solution u of (1.1) follows directly from our construction.

3.1 Proof of uniqueness for Theorem 2.2

Proof of Uniqueness for Theorem 2.2. Fix a stopping time $\tau \leq T$ and $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable random field φ such that for some $\beta' \in (\alpha, \bar{\beta} \wedge \tilde{\beta})$ and $\theta' \geq 0$, \mathbf{P} -a.s. $r_1^{-\theta'} \varphi \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$. In this section we will drop the dependence of processes t, x , and z when we feel it will not obscure the argument. Let $\hat{u}^1(\tau)$ and $\hat{u}^2(\tau)$ be solutions of (2.1) in $\mathfrak{U}^{\beta'}$. It follows that $v := \hat{u}^1(\tau) - \hat{u}^2(\tau)$ solves

$$\begin{aligned} dv_t^l &= [(\mathcal{L}_t^{1;l} + \mathcal{L}_t^{2;l})v_t + \hat{b}_t^i \partial_i v_t^l + \hat{c}_t^{\bar{l}} v_t^{\bar{l}}] dt + \mathcal{N}_t^{1;\varphi} v_t^l dw_t^{1;\varphi} \\ &\quad + \int_{Z^1} \mathcal{I}_{t,z}^{1;l} v_{t-} [\mathbf{1}_{D^1}(z) q^1(dt, dz) + \mathbf{1}_{E^1}(z) p^1(dt, dz)], \quad \tau < t \leq T, \\ v_t^l &= 0, \quad t \leq \tau, \quad l \in \{1, \dots, d_2\}, \end{aligned}$$

and \mathbf{P} -a.s.

$$\mathbf{1}_{[\tau_n, \tau_{n+1})} r_1^{-\lambda_n} v \in D([0, T]; C^{\beta'}(\mathbf{R}^{d_1}, \mathbf{R}^{d_2})),$$

where $(\tau_n)_{n \geq 0}$ is an increasing sequence of \mathbf{F} -stopping times with $\tau_0 = 0$ and $\tau_n = T$ for sufficiently large n , and where for each n , λ_n is a positive \mathcal{F}_{τ_n} -measurable random variable. Clearly it suffices to take $\tau_1 = \tau$ and $\lambda_0 = 0$. Thus, $v_t(x) = 0$ for all $(\omega, t) \in [[\tau_0, \tau_1))$. Assume that for some n , \mathbf{P} -a.s. for all t and x , $v_{t \wedge \tau_n}(x) = 0$. We will show that \mathbf{P} -a.s. for all t and x , $\tilde{v}_t(x) := v_{(\tau_n \vee t) \wedge \tau_{n+1}}(x) = 0$. Applying Itô's formula, for each x , \mathbf{P} -a.s. for all t , we find

$$\begin{aligned} d|\tilde{v}_t|^2 &= \left(2\tilde{v}_t^l \mathcal{L}_t^{1;l} \tilde{v}_t + |\mathcal{N}_t^1 \tilde{v}_t|^2 + 2\tilde{v}_t^l \hat{b}_t^i \partial_i \tilde{v}_t^l + 2\tilde{v}_t^l \hat{c}_t^{\bar{l}} \tilde{v}_t^{\bar{l}} \right) dt \\ &\quad + \left(2\tilde{v}_t^l \mathcal{I}_{t,z}^{1;l} \tilde{v}_t + \int_{D^1 \cup E^1} |\mathcal{I}_{t,z}^{1;l} \tilde{v}_t|^2 \pi^1(dz) \right) dt \\ &\quad + \left(2\tilde{v}_t^l \mathcal{L}_t^{2;l} \tilde{v}_t + 2\tilde{v}_t^l \mathcal{I}_{t,z}^{2;l} \tilde{v}_t \right) dt + 2\tilde{v}_t^l \mathcal{N}_t^{1;\varphi} \tilde{v}_t^l dw_t^{1;\varphi} \end{aligned}$$

$$\begin{aligned}
& + \int_{Z^1} \left(2\tilde{v}_{t-}^l \mathcal{I}_{t,z}^{1;l} \tilde{v}_{t-} + |\mathcal{I}_{t,z}^{1;l} \tilde{v}_{t-}|^2 \right) q^1(dt, dz), \quad \tau_n < t \leq \tau_{n+1}, \\
& |\tilde{v}_t|^2 = 0, \quad t \leq \tau_n, \quad l \in \{1, \dots, d_2\},
\end{aligned} \tag{3.1}$$

where for $\phi \in C_c^\infty(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$, $k \in \{1, 2\}$, and $l \in \{1, \dots, d_2\}$,

$$\mathfrak{Q}^{k;l} \phi := \frac{1}{2} \sigma^{k;i\varrho} \sigma^{k;j\varrho} \partial_{ij} \phi^l + \sigma^{k;j\varrho} \partial_j \sigma^{k;i\varrho} \partial_i \phi^l + \sigma^{k;i\varrho} \nu^{k;\bar{l}\varrho} \partial_i \phi^{\bar{l}} + \sigma^{k;j\varrho} \partial_j \alpha^{k;\bar{l}\varrho} \phi^{\bar{l}}$$

and

$$\begin{aligned}
\mathfrak{Z}^{k;l} \phi &:= \int_{D^k} \left(\rho^{k;\bar{l}\bar{l}} \phi^{\bar{l}}(\tilde{H}^k) - \rho^{k;\bar{l}\bar{l}}(\tilde{H}^{k;-1}) \phi^{\bar{l}} \right) \pi^k(dz) \\
&+ \int_{D^k} \left(\phi^l(\tilde{H}^k) - \phi^l + \mathbf{1}_{(1,2]}(\alpha) F^{k;i} \partial_i \phi^l \right) \pi^k(dz) \\
&+ \int_{E^k} \left((I_{d_2}^{\bar{l}\bar{l}} + \rho^{k;\bar{l}\bar{l}}) \phi^{\bar{l}}(\tilde{H}^k) - \phi^l \right) \pi^k(dz).
\end{aligned}$$

For each ω and t , let

$$Q_t = \int_{\mathbf{R}^{d_1}} |\tilde{v}_t(x)|^2 r_1^{-\lambda}(x) dx,$$

where $\lambda = \lambda_n + (d' + 2)/2$ and $d' > d_1$. Note that

$$\mathbf{E} Q_t \leq \int_{\mathbf{R}^{d_1}} r_1^{-d'}(x) dx \mathbf{E} |r_1^{-\lambda_n} \tilde{v}_t|_0 < \infty.$$

It suffices to show that $\sup_{t \leq T} \mathbf{E} Q_t = 0$. To this end, we will multiply the equation (3.1) by the weight $r_1^{-2\lambda} = r_1^{-2\lambda_n+1} r_1^{-d'}$, integrate in x , and change the order of the integrals in time and space. Thus, we must verify the assumptions of stochastic Fubini theorem hold (see Corollary 4.13 and Remark 4.14 as well) with the finite measure $\mu(dx) = r_1^{-d'}(x) dx$ on \mathbf{R}^{d_1} . Since b and σ^k have linear growth and ν^k and c are bounded, owing to Lemma 4.6, we easily obtain that there is a constant $N = N(d_1, d_2, N_0, \lambda_n)$ such that \mathbf{P} -a.s for all t ,

$$\int_{\mathbf{R}^{d_1}} \left(\sum_{k=1}^2 2|r_1^{-\lambda_n} \tilde{v}| |r_1^{-\lambda_n-2} \mathfrak{Q}^k \tilde{v}| + |r_1^{\lambda_n-1} \mathcal{N}^1 \tilde{v}|^2 \right) r_1^{-d'} dx \leq N \sup_{t \leq T} |r_1^{-\lambda_n} \tilde{v}_t|_{\beta'}^2,$$

$$\int_{\mathbf{R}^{d_1}} 4|r_1^{-\lambda_n} \tilde{v}|^2 |r_1^{-\lambda_n-1} \mathcal{N}^1 \tilde{v}|^2 r_1^{-d'} dx \leq N \sup_{t \leq T} |r_1^{-\lambda_n} \tilde{v}_t|_{\beta'}^4,$$

and

$$\int_{\mathbf{R}^{d_1}} \left(2|r_1^{-\lambda_n} \tilde{v}| |r_1^{-\lambda_n-1} b \partial_i \tilde{v}| + 2|r_1^{-\lambda_n} \tilde{v}| |r_1^{-\lambda_n} c \tilde{v}| \right) r_1^{-d'} dx \leq N \sup_{t \leq T} |r_1^{-\lambda_n} \tilde{v}_t|_{\beta'}^2.$$

For all $\phi \in C_{loc}^\alpha(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ and all k, ω, t, x, p , and z ,

$$\begin{aligned} & r_1^{-p}(\phi(\tilde{H}^k) - \phi + \mathbf{1}_{(1,2]}(\alpha)F^{k;i}\partial_i\phi) \\ &= \bar{\phi}(\tilde{H}^k) - \bar{\phi} - \mathbf{1}_{(1,2]}(\alpha)H^{k;i}\partial_i\bar{\phi} + \mathbf{1}_{(1,2]}(\alpha)(H^{k;i} + F^{k;i})\partial_i\bar{\phi} \\ &+ p\mathbf{1}_{(1,2]}(\alpha)(H^{k;i} + F^{k;i})r_1^{-2}x^i\bar{\phi} + \left(\frac{r_1^p(\tilde{H}^k)}{r_1^p} - 1\right)(\bar{\phi}(\tilde{H}^k) - \mathbf{1}_{(1,2]}(\alpha)\bar{\phi}) \\ &+ \mathbf{1}_{(1,2]}(\alpha)\left(\frac{r_1^p(\tilde{H}^k)}{r_1^p} - 1 + pH^{k;i}r_1^{-2}x^i\right)\bar{\phi}, \end{aligned} \quad (3.2)$$

where $\bar{\phi} := r^{-p}\phi$. By Taylor's formula, for all $\phi \in C^\alpha(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ and all k, ω, t, x , and z , we have

$$|\phi(\tilde{H}^k) - \phi - \mathbf{1}_{(1,2]}(\alpha)H^{k;i}\partial_i\phi| \leq r_1^\alpha|\phi|_\alpha|r_1^{-1}H|_0^\alpha. \quad (3.3)$$

Combining (3.2), (3.3), and the estimates given in Lemma 4.10 (1), for all k, ω, t, x and z , we obtain

$$r_1^{-\alpha}|\rho^k(\tilde{H}^{k;-1}) - \rho^k| \leq N|\rho|_{\alpha \wedge 1}|r_1^{-1}H^k|_0^{\alpha \wedge 1}$$

and

$$\begin{aligned} & r_1^{-\lambda_n - \alpha}|\tilde{v}(\tilde{H}^k) - \tilde{v} + \mathbf{1}_{(1,2]}(\alpha)F^{k;i}\partial_i\tilde{v}| \\ & \leq N|r_1^{-\lambda_n}\tilde{v}|_\alpha \left(|r_1^{-1}H^k|_0^\alpha + |r_1^{-1}H|_0[H^k]_1 + |r_1^{-1}H|_0^{[\alpha]^-+1} + [H]_1^{[\alpha]^-+1}\right), \end{aligned} \quad (3.4)$$

for some constant $N = N(d_1, \lambda_n, N_0, \eta^1, \eta^2)$. Therefore, \mathbf{P} -a.s for all t ,

$$\int_{\mathbf{R}^{d_1}} \left(\sum_{k=1}^2 2|r_1^{-\lambda_n}\tilde{v}||r_1^{-\lambda_n-2}\mathfrak{I}^k\tilde{v}| + \int_{D^1 \cup E^1} |r_1^{-\lambda-1}\mathcal{I}_z\tilde{v}|^2\pi^1(dz) \right) r_1^{-d'}dx \leq N \sup_{t \leq T} |r_1^{-\lambda_n}\tilde{v}|_{\beta'}^2,$$

and

$$\int_{\mathbf{R}^{d_1}} \left(2|r_1^{-\lambda_n}\tilde{v}||r_1^{-\lambda_n-2}\mathcal{I}_z^k\tilde{v}| + |r_1^{-\lambda_n-1}\mathcal{I}_z\tilde{v}|^2 \right) r_1^{-d'}dx \leq N \sup_{t \leq T} |r_1^{-\lambda_n}\tilde{v}|_{\beta'}^4,$$

for some constant $N = N(d_1, d_2, \lambda_n, N_0, \eta^1, \eta^2)$.

Let $L^2(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ be the space of square-integrable functions $f : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_2}$ with norm $\|\cdot\|_0$ and inner product $(\cdot, \cdot)_0$. Moreover, let $L^2(\mathbf{R}^{d_1}; \ell_2(\mathbf{R}^{d_2}))$ be the space of square-integrable functions $f : \mathbf{R}^{d_1} \rightarrow \ell_2(\mathbf{R}^{d_2})$ with norm $\|\cdot\|_0$. With the help of the above estimates and Corollary 4.13, denoting $\bar{v} = r^{-\lambda}\tilde{v}$, \mathbf{P} -a.s. for all t , we have

$$\begin{aligned} d\|\bar{v}_t\|_0^2 &= \left(2(\bar{v}_t^l, \bar{\mathcal{Q}}_t^1\bar{v}_t)_0 + \|\bar{\mathcal{N}}_t^1\bar{v}_t\|_0^2 + 2(\bar{v}_t, \bar{\mathcal{S}}_{t,z}^1\bar{v}_t)_0 + \int_{D^1 \cup E^1} \|\bar{\mathcal{I}}_{t,z}^1\bar{v}_t\|_0^2\pi^1(dz) \right) dt \\ &+ \left(2(\bar{v}_t, b_t^i\partial_i\bar{v}_t + \bar{c}_t^{\bar{l}}\bar{v}_t^{\bar{l}})_0 + 2(\bar{v}_t, \bar{\mathcal{Q}}_t^2\bar{v}_t)_0 + 2(\bar{v}_t, \bar{\mathcal{S}}_{t,z}^2\bar{v}_t)_0 \right) dt + 2(v_t, \bar{\mathcal{N}}_t^{1;q}\bar{v}_t)_0 dw_t^{1;q} \\ &+ \int_{Z^1} \left(2(\bar{v}_{t-}, \bar{\mathcal{I}}_{t,z}^1\bar{v}_{t-})_0 + \|\bar{\mathcal{I}}_{t,z}^1\bar{v}_{t-}\|_0^2 \right) q^1(dt, dz), \quad \tau_n < t \leq \tau_{n+1}, \\ \|\bar{v}_t\|_0^2 &= 0, \quad t \leq \tau_n, \quad l \in \{1, \dots, d_2\}, \end{aligned} \quad (3.5)$$

where all coefficients and operators are defined as in (2.1) with the following changes:

(1) for each $k \in \{1, 2\}$, v^k is replaced with

$$\bar{v}^{k;\bar{l}\bar{l}} := v^{k;\bar{l}\bar{l}} + \mathbf{1}_{\{2\}}(\alpha) \lambda \sigma^{k;i\bar{e}} r_1^{-2} x^i \delta_{\bar{l}\bar{l}};$$

(2) for each $k \in \{1, 2\}$, ρ^k replaced with

$$\bar{\rho}^{k;\bar{l}\bar{l}} := \rho^{k;\bar{l}\bar{l}} + \left(\frac{r_1^\lambda(\tilde{H}^k)}{r_1^\lambda} - 1 \right) (I_{d_2}^{\bar{l}\bar{l}} + \rho^{k;\bar{l}\bar{l}});$$

(3) c is replaced with

$$\begin{aligned} \bar{c}^{\bar{l}\bar{l}} &= c^{\bar{l}\bar{l}} + \lambda b^i r^{-2} x^i \delta_{\bar{l}\bar{l}} + \sum_{k=1}^2 \lambda^2 \sigma^{k;i\bar{e}} \sigma^{k;j\bar{e}} r_1^{-4} x^i x^j \\ &+ \sum_{k=1}^2 \int_{D^k} \left(\left(\frac{r_1^\lambda}{r_1^\lambda(\tilde{H}^{k;-1})} - 1 \right) (I_m^{\bar{l}\bar{l}} + \rho^k(\tilde{H}^{k;-1})) - \mathbf{1}_{\{1,2\}}(\alpha) \lambda r_1^{-2} x_i H^{k;i}(\tilde{H}^{k;-1}) \right) \pi^k(dz). \end{aligned}$$

Since for all k, ω and t , $|r_1^{-1} \sigma^k|_0 + |r_1^{-1} \nabla \sigma^k|_{\bar{\beta}-1} + |v^k|_{\bar{\beta}} \leq N_0$, for $\bar{\beta} > 1 \vee \alpha$ and $\bar{\beta} > \alpha$, it is clear that $|\bar{v}^k|_\alpha \leq N$. Moreover, since for all k, ω and t , $|r_1^{-1} H^k|_0 + |H^k|_{\bar{\beta}} \leq K^k$ and $|\rho|_{\bar{\beta}'} \leq l^k$, applying the estimates in Lemma (4.10) (1), we get

$$|\bar{\rho}^k|_\alpha \leq l^k + K^k(1 + l^k) \quad \text{and} \quad |c|_\alpha \leq N_0.$$

We will now estimate the drift terms of (3.5) in terms of $\|\bar{v}_t\|_0^2$. We write $f \sim g$ if $\int_{\mathbf{R}^{d_1}} |f(x)| dx = \int_{\mathbf{R}^{d_1}} |g(x)| dx$ and $f \ll g$ if $\int_{\mathbf{R}^{d_1}} |f(x)| dx \leq \int_{\mathbf{R}^{d_1}} |g(x)| dx$. Using the divergence theorem, for any $v : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_2}$, $\sigma : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_1}$ and $v : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{2d_2}$ and all x , we get

$$\begin{aligned} \sigma^i \sigma^j v^l v_{ij}^l &\sim \frac{1}{2} (\sigma^i \sigma^j)_{ij} v - \sigma^i \sigma^j v_i^l v_j^l = (\sigma_{ij}^i \sigma^j + \sigma_{ij}^j \sigma_i^j) |v|^2 - \sigma^i \sigma^j v_i^l v_j^l, \\ 2\sigma_{ij}^i \sigma^j v^l v_i^l &\sim -(\sigma_{ij}^i \sigma^j)_i |v|^2 = (\sigma_{ij}^i \sigma^j + \sigma_{ij}^j \sigma_i^j) |v|^2, \end{aligned}$$

and

$$\sigma^i v^l v_{ij}^l v_i^l + \sigma^i v^l v_{ij}^l v_i^l = \sigma^i v^l v_{sym}^l v_i^l \sim -(\sigma^i v_{sym}^l)_i |v|^2 = -(\sigma_i^i v_{sym}^l + \sigma^i v_{sym}^l) |v|^2,$$

where $v_{sym}^l = (v^l + v^l)/2$. Consequently, for all ω, t , and x , we have

$$2\bar{v}^l \bar{\mathcal{Q}}^{1;l} \bar{v} + |\bar{\mathcal{N}}^1 \bar{v}|^2 \sim \frac{1}{2} \left(|\operatorname{div} \sigma^1|^2 - \partial_i \sigma^{1;j\bar{e}} \partial_j \sigma^{1;i\bar{e}} \right) |\bar{v}|^2 - \bar{v}_{sym}^{1;l\bar{e}} \bar{v}^l \bar{v}^{\bar{l}} \operatorname{div} \sigma^{1;\bar{e}} + |\bar{v}^1 \bar{v}|^2 \ll N |\bar{v}|^2$$

and

$$2\bar{v}^l \bar{\mathcal{Q}}^{(2);l} \bar{v} \ll -(1 + \epsilon) |\sigma^{2;i} \partial_i \bar{v}|^2 + N |\bar{v}|^2,$$

for any $\epsilon > 0$, where in the last estimate we have also used Young's inequality. By Lemma 4.10 (2) and basic properties of the determinant, there is a constant $N = N(d, N_0, \eta^1, \eta^2)$ such that for all k, ω, t, x , and z ,

$$\det \tilde{H}^{k;-1} - 1 = \det(I_d + F^k) - 1 \leq |\nabla F^k| \leq N |\nabla H^k|$$

and

$$\det \tilde{H}^{k;-1} - 1 - \operatorname{div} F^k \leq |\nabla F^k|^2 \leq N |\nabla H^k|^2.$$

Thus, integrating by parts, for all ω, t , and x , we get

$$\begin{aligned} 2\bar{v}^l \tilde{\mathfrak{S}}^{1;l} \bar{v} + \int_{D^1 \cup E^1} |\tilde{T}^1 \bar{v}|^2 \pi^1(dz) &\sim 2 \int_{D^1} \bar{\rho}_{sym}^{1;l}(\tilde{H}^{1;-1}) (\det \nabla \tilde{H}^{1;-1} - 1) \pi^1(dz) \bar{v}^l \bar{v}^l \\ &+ \int_{D^1 \cup E^1} \left(\det \nabla \tilde{H}^{1;-1} - 1 + \mathbf{1}_{(1,2]}(\alpha) \mathbf{1}_{D^1} \operatorname{div} F^1 \right) \pi^1(dz) |\bar{v}|^2 \\ &+ \int_{D^1 \cup E^1} \left(\mathbf{1}_{E^1} 2\bar{\rho}_{sym}^{1;l}(\tilde{H}^{1;-1}) \bar{v}^l \bar{v}^l + |\bar{\rho}^1(\tilde{H}^{1;-1}) \bar{v}|^2 \right) \det \nabla \tilde{H}^{1;-1} \pi^1(dz) \\ &\ll N \left(\int_{D^1} \left(K^1(z)^2 + l^1(z) K^1(z) + l^1(z)^2 \right) \pi^1(dz) + \int_{E^1} \left(K^k(z) + l^k(z) \right) \pi^1(dz) \right) |\bar{v}|^2. \end{aligned}$$

Analogously, for all ω, t , and x , we obtain

$$2\bar{v}^l \tilde{\mathfrak{S}}^{2;l} \bar{v} \leq -(1 + \epsilon) \int_{D^2 \cup E^2} |\bar{v}(\tilde{H}^2) - \bar{v}|^2 \pi^2(dz) + N |\bar{v}|^2.$$

Therefore, combining the above estimates, **P**-a.s. for all t ,

$$Q_t \leq N \int_0^t Q_s ds + M_t, \quad (3.6)$$

where $(M_t)_{t \leq T}$ is a càdlàg square-integrable martingale. Taking the expectation of (3.6) and applying Gronwall's lemma, we get $\sup_{t \leq T} \mathbf{E} Q_t = 0$, which implies that **P**-a.s. for all t and x , $\tilde{v}_t(x) = 0$. This completes the proof. \square

3.2 Small jump case

Set $(w^e)_{e \geq 1} = (w^{1:e})_{e \geq 1}$, $(Z, \mathcal{Z}, \pi) = (\mathcal{Z}^1, \mathcal{Z}^1, \pi^1)$, $p(dt, dz) = p^1(dt, dz)$, and $q(dt, dz) = q^1(dt, dz)$. Let $\sigma_t(x) = (\sigma_t^{i0}(x))_{1 \leq i \leq d_1, e \geq 1}$ be a $\ell_2(\mathbf{R}^{d_1})$ -valued $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable function defined on $\Omega \times [0, T] \times \mathbf{R}^{d_1}$ and $H_t(x, z) = (H_t^i(x, z))_{1 \leq i \leq d_1}$ be a $\mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}$ -measurable function defined on $\Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z$.

We introduce the following assumption for $\beta > 1 \vee \alpha$.

Assumption 3.1 (β). (1) There is a constant $N_0 > 0$ such that for all $(\omega, t) \in \Omega \times [0, T]$,

$$|r_1^{-1} b_t|_0 + |r_1^{-1} \sigma_t|_0 + |\nabla b_t|_{\beta-1} + |\nabla \sigma_t|_{\beta-1} \leq N_0.$$

Moreover, for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$,

$$|r_1^{-1} H_t(z)|_0 \leq K_t(z) \quad \text{and} \quad |\nabla H_t(z)|_{\beta-1} \leq \bar{K}_t(z),$$

where $K : \Omega \times [0, T] \times Z \rightarrow \mathbf{R}_+$ is a $\mathcal{P}_T \otimes \mathcal{Z}$ -measurable function satisfying

$$K_t(z) + \bar{K}_t(z) + \int_Z \left(K_t(z)^\alpha + \bar{K}_t(z)^2 \right) \pi(dz) \leq N_0,$$

for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$.

(2) *There is a constant $\eta \in (0, 1)$ such that for all $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z : |\nabla H_t(\omega, x, z)| > \eta\}$,*

$$|(I_{d_1} + \nabla H_t(x, z))^{-1}| \leq N_0.$$

Let Assumption 3.1(β) hold for some $\beta > 1 \vee \alpha$. Let $\tau \leq T$ be a stopping time. Consider the system of SDEs on $[0, T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned} dv_t(x) = & \left(\mathbf{1}_{\{2\}}(\alpha) \frac{1}{2} \sigma_t^{i\varrho}(x) \sigma_t^{j\varrho}(x) \partial_{ij} v_t(x) + b_t^i(x) \partial_i v_t(x) \right) dt + \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{i\varrho}(x) \partial_i v_t(x) dw_t^{\varrho} \\ & + \mathbf{1}_{\{1,2\}}(\alpha) \int_Z (v_t(x + H_t(x, z)) - v_t(x) + F_t(x, z) \partial_i v_t(x)) \pi(dz) dt \\ & + \int_Z (v_{t-}(x + H_t(x, z)) - v_{t-}(x)) [\mathbf{1}_{\{1,2\}}(\alpha) q(dt, dz) + \mathbf{1}_{\{0,1\}}(\alpha) p(dt, dz)], \quad \tau < t \leq T, \\ v_t(x) = & x, \quad t \leq \tau, \end{aligned} \quad (3.7)$$

where

$$b_t^i(x) := \mathbf{1}_{\{1,2\}}(\alpha) b_t^i(x) + \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{i\varrho}(x) \partial_j \sigma_t^{j\varrho}(x)$$

and

$$F_t(x, z) := -H_t(\tilde{H}_t^{-1}(x, z), z).$$

We associate with (3.7), the stochastic flow $Y_t = Y_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, generated by the SDE

$$\begin{aligned} dY_t = & -\mathbf{1}_{\{1,2\}}(\alpha) b_t(Y_t) dt - \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{\varrho}(Y_t) dw_t^{\varrho} \\ & + \int_Z F_t(Y_{t-}, z) [\mathbf{1}_{\{1,2\}}(z) q(dt, dz) + \mathbf{1}_{\{0,1\}}(z) p(dt, dz)], \quad \tau < t \leq T, \\ Y_t = & x, \quad t \leq \tau. \end{aligned} \quad (3.8)$$

Owing to parts (1) and (2) of Lemma 4.10, for each ω, t , and z , the inverse of the mapping $\tilde{F}_t(x, z) := x + F_t(x, z) = x - H_t(\tilde{H}_t^{-1}(x, z), z)$ is $\tilde{H}_t(x, z) := x + H_t(x, z)$ and there is a constant $N = N(d_1, N_0, \beta, \eta)$ such that for all ω, t, x, y , and z ,

$$|r_1^{-1} F_t(z)|_0 \leq N K_t(z), \quad |\nabla F_t(z)|_{\beta-1} \leq K_t(z), \quad |(I_{d_1} + \nabla F_t(x, z))^{-1}| \leq N.$$

Thus, by Theorem 2.1 in [LM14b], there is a modification of the solution of (3.8), which we still denote by $Y_t = Y_t(\tau, x)$, that is a $C_{loc}^{\beta'}$ -diffeomorphism for any $\beta' \in [1, \beta)$. Moreover, \mathbf{P} -a.s. $Y(\tau, \cdot), Y^{-1}(\tau, \cdot) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_1}))$, and $Y_{t-}^{-1}(\tau, \cdot)$ coincides with the inverse of $Y_{t-}(\tau, \cdot)$ for all t . The following proposition shows that the inverse flow $Y_t^{-1}(\tau)$ solves (3.7).

Proposition 3.1. *Let Assumption 3.1(β) hold for some $\beta > 1 \vee \alpha$. For each stopping time $\tau \leq T$ and $\beta' \in [1 \vee \alpha, \beta)$, $v_t(x) = v_t(\tau, x) = Y_t^{-1}(\tau, x)$ solves (3.7) and for each $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d_1, p, N_0, T, \beta', \eta, \epsilon)$ such that*

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)} v_t(\tau)|_0^p \right] + \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla v_t(\tau)|_{\beta'-1}^p \right] \leq N. \quad (3.9)$$

Proof. The estimate (3.9) is given in Theorem 2.1 in [LM14b] (see also Remark 2.1), so we only need to show that $Y_t^{-1}(\tau, x)$ solves (3.7). Let $(\delta_n)_{n \geq 1}$ be a sequence such that $\delta_n \in (0, \eta)$ for all n and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. It is clear that there is a constant $N = N(N_0)$ such that for all ω and t ,

$$\pi(\{z : K_t(z) > \delta_n\}) \leq \frac{N}{\delta_n^\alpha}. \quad (3.10)$$

For each n , consider the system of SDEs on $[0, T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned} dv_t^{(n)}(x) = & \left(\mathbf{1}_{\{2\}}(\alpha) \frac{1}{2} \sigma_t^{i\varrho}(x) \sigma_t^{j\varrho}(x) \partial_{ij} v_t^{(n)}(x) + b_t^i(x) \partial_i v_t^{(n)}(x) \right) dt \\ & + \mathbf{1}_{(1,2]}(\alpha) \int_Z \mathbf{1}_{\{K_t > \delta_n\}}(z) \left(v_t^{(n)}(x + H_t(x, z)) - v_t^{(n)}(x) + F_t^i(x, z) \partial_i v_t^{(n)}(x) \right) \pi(dz) dt \\ & + \int_Z \mathbf{1}_{\{K_t > \delta_n\}}(z) \left(v_{t-}^{(n)}(x + H_t(x, z)) - v_{t-}^{(n)}(x) \right) [\mathbf{1}_{(1,2]}(\alpha) q(dt, dz) + \mathbf{1}_{[0,1]}(\alpha) p(dt, dz)], \\ & + \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{i\varrho}(x) \partial_i v_t^{(n)}(x) dw_t^\varrho, \quad \tau < t \leq T, \quad v_t^{(n)}(x) = x, \quad t \leq \tau, \end{aligned} \quad (3.11)$$

and the stochastic flow $Y_t^{(n)} = Y_t^{(n)}(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, generated by the SDE

$$\begin{aligned} dY_t^{(n)} = & -\mathbf{1}_{[1,2]}(\alpha) b_t(Y_t^{(n)}) dt - \mathbf{1}_{\{2\}}(\alpha) \sigma_t^\varrho(Y_t^{(n)}) dw_t^\varrho \\ & + \int_Z \mathbf{1}_{\{K_t > \delta_n\}}(z) F_t(Y_{t-}^{(n)}, z) [\mathbf{1}_{(1,2]}(\alpha) q(dt, dz) + \mathbf{1}_{[0,1]}(\alpha) p(dt, dz)], \quad \tau < t \leq T, \\ Y_t^{(n)}(x) = & x, \quad t \leq \tau. \end{aligned} \quad (3.12)$$

Since (3.10) holds, we can rewrite equation (3.12) as

$$\begin{aligned} dY_t^{(n)} = & -\left(\mathbf{1}_{[1,2]}(\alpha) b_t(Y_t^{(n)}) + \mathbf{1}_{(1,2]}(\alpha) \int_Z \mathbf{1}_{\{K_t > \delta_n\}}(z) F_t(Y_t^{(n)}, z) \pi(dz) \right) dt \\ & - \mathbf{1}_{\{2\}}(\alpha) \sigma_t^\varrho(Y_t^{(n)}) dw_t^\varrho + \int_Z \mathbf{1}_{\{K_t > \delta_n\}}(z) F_t(Y_{t-}^{(n)}, z) p(dt, dz), \quad \tau < t \leq T, \end{aligned} \quad (3.13)$$

and (3.11) as

$$\begin{aligned} dv_t^{(n)}(x) = & \left(\mathbf{1}_{\{2\}}(\alpha) \frac{1}{2} \sigma_t^{i\varrho}(x) \sigma_t^{j\varrho}(x) \partial_{ij} v_t^{(n)}(x) + b_t^i(x) \partial_i v_t^{(n)}(x) \right) dt \\ & + \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{i\varrho}(x) \partial_i v_t^{(n)}(x) dw_t^\varrho + \mathbf{1}_{(1,2]}(\alpha) \int_Z \mathbf{1}_{\{K_t > \delta_n\}}(z) F_t^i(x, z) \pi(dz) \partial_i v_t^{(n)}(x) dt \\ & + \int_Z \mathbf{1}_{\{K_t > \delta_n\}}(z) \left(v_{t-}^{(n)}(x + H_t(x, z)) - v_{t-}^{(n)}(x) \right) p(dt, dz), \quad \tau < t \leq T. \end{aligned} \quad (3.14)$$

We claim that the solution $Y_t^{(n)} = Y_t^{(n)}(x)$ of (3.13) can be written as the solution of continuous SDEs with a finite number of jumps interlaced. Indeed, for each n and stopping time $\tau' \leq T$,

consider the stochastic flow $\tilde{Y}_t^{(n)} = \tilde{Y}_t^{(n)}(\tau', x)$, $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, generated by the SDE

$$\begin{aligned} d\tilde{Y}_t^{(n)} &= -[\mathbf{1}_{[1,2]}(\alpha)b_t(\tilde{Y}_t^{(n)}) + \mathbf{1}_{(1,2]}(\alpha) \int_Z \mathbf{1}_{\{K>\delta_n\}}(t, z)F_t(\tilde{Y}_t^{(n)}, z)\pi(dz)]dt \\ &\quad - \mathbf{1}_{\{2\}}(\alpha)\sigma_t^o(\tilde{Y}_t^{(n)})dw_t^o, \quad \tau' < t \leq T, \\ \tilde{Y}_t^{(n)} &= x, \quad t \leq \tau'. \end{aligned}$$

By Theorems 2.1 and 2.4 and Remark 2.2 in [LM14b], there is a modification of $\tilde{Y}_t^{(n)} = \tilde{Y}_t^{(n)}(\tau', x)$, still denoted $\tilde{Y}_t^{(n)}(\tau', x)$, that is a $C_{loc}^{\beta'}$ -diffeomorphism. Furthermore, \mathbf{P} -a.s. we have that

$$\tilde{Y}_\cdot^{(n)}(\tau', \cdot), \tilde{Y}_\cdot^{(n);-1}(\tau', \cdot) \in C([0, T]; C_{loc}^{\beta'})$$

and $\tilde{v}_t^{(n)} = \tilde{v}_t^{(n)}(\tau', x) = \tilde{Y}_t^{(n);-1}(\tau', x)$ solves the SPDE given by

$$\begin{aligned} d\tilde{v}_t^{(n)}(x) &= \left(\mathbf{1}_{\{2\}}(\alpha) \frac{1}{2} \sigma_t^{io}(x) \sigma_t^{jo}(x) \partial_{ij} v_t^{(n)}(x) + b_t^i(x) \partial_i v_t^{(n)}(x) \right) dt \\ &\quad + \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{io}(x) \partial_i v_t^{(n)}(x) dw_t^o \\ &\quad + \mathbf{1}_{(1,2]}(\alpha) \int_Z \mathbf{1}_{\{K>\delta_n\}}(t, z) F_t^i(t, z) \pi(dz) dt \partial_i v_t^{(n)}(x), \quad \tau' < t \leq T, \\ \tilde{v}_t^{(n)}(x) &= x, \quad t \leq \tau'. \end{aligned}$$

For each n , let

$$A_t^{(n)} = \int_{[0,t]} \int_Z \mathbf{1}_{\{K_s>\delta_n\}}(z) p(ds, dz), \quad t \geq 0,$$

and define the sequence of stopping times $(\tau_l^{(n)})_{l=1}^\infty$ recursively by $\tau_0^{(n)} = \tau$ and

$$\tau_{l+1}^{(n)} = \inf \{t > \tau_l^{(n)} : \Delta A_t^{(n)} \neq 0\} \wedge T.$$

Fix some $n \geq 1$. It is clear that \mathbf{P} -a.s. for all x and $t \in [0, \tau_1^{(n)})$,

$$Y_t^{(n);-1}(\tau, x) = \tilde{Y}_t^{(n);-1}(\tau, x) = \tilde{v}_t^{(n)}(\tau, x)$$

satisfies (3.14) up to, but not including time $\tau_1^{(n)}$. Moreover, \mathbf{P} -a.s. for all x ,

$$Y_{\tau_1^{(n)}}^{(n)}(\tau, x) = \tilde{Y}_{\tau_1^{(n)}}^{(n)}(\tau, x) + \int_Z F_{\tau_1^{(n)}}(\tilde{Y}_{\tau_1^{(n)}}^{(n)}(\tau, x), z) p(\{\tau_1^{(n)}\}, dz),$$

and hence

$$Y_{\tau_1^{(n)}}^{(n);-1}(\tau, x) = \int_Z \tilde{v}_{\tau_1^{(n)}}^{(n)}(\tau, x + H_{\tau_1^{(n)}}(x, z)) p(\{\tau_1^{(n)}\}, dz).$$

Consequently, $v_t^{(n)}(\tau, x) = Y_t^{(n);-1}(\tau, x)$ solves (3.14) up to and including time $\tau_1^{(n)}$. Assume that for some $l \geq 1$, $v_t^{(n)}(\tau, x) = Y_t^{(n);-1}(\tau, x)$ solves (3.14) up to and including time $\tau_l^{(n)}$. Clearly,

P-a.s. for all x and $t \in [\tau_l^{(n)}, \tau_{l+1}^{(n)})$, $Y_t^{(n)}(x) = \tilde{Y}_t^{(n)}(\tau_l^{(n)}, Y_{\tau_l^{(n)}-}^{(n)}(x))$, and thus **P**-a.s. for all x and $t \in [\tau_l^{(n)}, \tau_{l+1}^{(n)})$,

$$Y_t^{(n);-1}(x) = \tilde{Y}_t^{(n)}(\tau_l^{(n)}, Y_{\tau_l^{(n)}-}^{(n)}(x)) = \tilde{v}_t^{(n)}(\tau_l^{(n)}, Y_{\tau_l^{(n)}-}^{(n)}(x)).$$

Moreover, **P**-a.s. for all x ,

$$Y_n^{-1}(\tau_{l+1}^n, x) = \int_U \tilde{v}_n(\tau_l^n, \tau_{l+1}^n-, x + H(\tau_{l+1}^n, x, z)) p(\{\tau_{l+1}^n\}, dz),$$

which implies that $v_t^{(n)}(\tau, x) = Y_t^{(n);-1}(\tau, x)$ solves (3.14) up to and including time τ_{l+1}^n . Therefore, by induction, for each n , $v_t^{(n)}(\tau, x) = Y_t^{(n);-1}(\tau, x)$ solves (3.14). It is easy to see that for all ω, t , and z ,

$$|r_1^{-1} \mathbf{1}_{\{K_t > \delta_n\}}(z) F_t(z) - r_1^{-1} F_t(z)|_0 + |\mathbf{1}_{\{K_t > \delta_n\}}(z) \nabla F_t(z) - \nabla F_t(z)|_{\beta-1} \leq \mathbf{1}_{\{K_t \leq \delta_n\}}(z) K_t(z)$$

and thus

$$d\mathbf{P}dt - \lim_{n \rightarrow \infty} \int_D \mathbf{1}_{\{K \leq \delta_n\}}(t, z) K_t(z)^2 \pi(dz) + d\mathbf{P}dt - \lim_{n \rightarrow \infty} \int_E \mathbf{1}_{\{K \leq \delta_n\}}(t, z) K_t(z) \pi(dz) = 0.$$

By virtue of Theorem 2.3 in [LM14b], for each $\epsilon > 0$, and $p \geq 2$, we have

$$\lim_{n \rightarrow \infty} \left(\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)}(Y_t^{(n)}(\tau) - r_1^{-(1+\epsilon)} Y_t(\tau))|_0^p \right] + \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla Y_t^{(n)}(\tau) - r_1^{-\epsilon} \nabla Y_t(\tau)|_{\beta'-1}^p \right] \right) = 0,$$

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)} Y_t^{(n);-1}(\tau) - r_1^{-(1+\epsilon)} Y_t^{-1}(\tau)|_0^p \right] = 0$$

and

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla Y_t^{(n);-1}(\tau) - r_1^{-\epsilon} \nabla Y_t^{-1}(\tau)|_{\beta'-1}^p \right] = 0.$$

Then passing to the limit in both sides of (3.11) and making use of Assumption 3.1(β), the estimate (3.4), and basic convergence properties of stochastic integrals, we find that $v_t(\tau, x) = X_t^{-1}(\tau, x)$ solves (3.7). \square

3.3 Adding free and zero-order terms

Set $(w^{\varrho})_{\varrho \geq 1} = (w^{1;\varrho})_{\varrho \geq 1}$, $(Z, \mathcal{Z}, \pi) = (\mathcal{Z}^1, \mathcal{Z}^1, \pi^1)$, $p(dt, dz) = p^1(dt, dz)$, and $q(dt, dz) = p^1(dt, dz) - \pi^1(dz)dt$. Also, set $D = D^1$, $E = E^1$, and assume $Z = D \cup E$. Let $v_t(x) = (v_t^{\bar{l}\varrho}(\omega, x))_{1 \leq l, \bar{l} \leq d_2, \varrho \geq 1}$ be a $\ell_2(\mathbf{R}^{2d_2})$ -valued $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable function defined on $\Omega \times [0, T] \times \mathbf{R}^{d_1}$ and $\rho_t(x, z) = (\rho_t^{\bar{l}}(\omega, x, z))_{1 \leq l, \bar{l} \leq d_2}$ be a $\mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}$ -measurable function defined on $\Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z$.

We introduce the following assumptions for $\beta > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$.

Assumption 3.2 (β). (1) There is a constant $N_0 > 0$ such that for all $(\omega, t) \in \Omega \times [0, T]$,

$$|r_1^{-1}b_t|_0 + |r_1^{-1}\sigma_t|_0 + |\nabla b_t|_{\beta-1} + |\nabla \sigma_t|_{\beta-1} \leq N_0.$$

Moreover, for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$,

$$|r_1^{-1}H_t(z)|_0 \leq K_t(z) \quad \text{and} \quad |\nabla H_t(z)|_{\beta-1} \leq \bar{K}_t(z),$$

where $K : \Omega \times [0, T] \times Z \rightarrow \mathbf{R}_+$ is a $\mathcal{P}_T \otimes \mathcal{Z}$ -measurable function satisfying

$$K_t(z) + \bar{K}_t(z) + \int_D (K_t(z)^\alpha + \bar{K}_t(z)^2) \pi(dz) + \int_E (K_t(z)^{\alpha \wedge 1} + \bar{K}_t(z)) \pi(dz) \leq N_0,$$

for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$.

(2) There is a constant $\eta \in (0, 1)$ such that for all $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z : |\nabla H_t(\omega, x, z)| > \eta\}$,

$$|(I_{d_1} + \nabla H_t(x, z))^{-1}| \leq N_0.$$

Assumption 3.3 ($\tilde{\beta}$). There is a constant $N_0 > 0$ such that for all $(\omega, t) \in \Omega \times [0, T]$,

$$|c_t|_{\tilde{\beta}} + |v_t|_{\tilde{\beta}} + |r_1^{-\theta} f_t|_{\tilde{\beta}} + |r_1^{-\theta} g_t|_{\tilde{\beta}} \leq N_0.$$

Moreover, for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$,

$$|\rho_t(z)|_{\tilde{\beta}} + |r_1^{-\theta} h_t(z)|_{\tilde{\beta}} \leq l_t(z),$$

where $l : \Omega \times [0, T] \times Z \rightarrow \mathbf{R}_+$ is a $\mathcal{P}_T \otimes \mathcal{Z}$ -measurable function satisfying

$$l_t(z) + \int_D l_t(z)^2 \pi(dz) + \int_E l_t(z) \pi(dz) \leq N_0.$$

$(\omega, t, z) \in \Omega \times [0, T] \times Z$.

Let Assumptions 3.2($\tilde{\beta}$) and 3.3($\tilde{\beta}$) hold for some $\tilde{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$. Let $\tau \leq T$ be a stopping time and $\varphi : \Omega \times \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_2}$ be a $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable random field. Consider the system of SDEs on $[0, T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned} dv_t^l &= \left(\mathcal{L}_t^l v_t + \hat{b}_t^l \partial_i \phi^l + \hat{c}_t^{\bar{l}} \phi^{\bar{l}} + \hat{f}_t^l \right) dt + \left(\mathcal{N}_t^{l\varrho} v_t + g_t^{l\varrho} \right) dw_t^{\varrho} \\ &\quad + \int_Z \left(I_{t,z}^l v_{t-} + h_t^l(z) \right) [\mathbf{1}_D(z) q(dt, dz) + \mathbf{1}_E(z) p(dt, dz)], \quad \tau < t \leq T, \\ v_t^l &= \phi^l, \quad t \leq \tau, \quad l \in \{1, \dots, d_2\}, \end{aligned} \tag{3.15}$$

where for $\phi \in C_c^\infty(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ and $l \in \{1, \dots, d_2\}$,

$$\begin{aligned} \mathcal{L}_t^l \phi(x) &:= \mathbf{1}_{\{2\}}(\alpha) \frac{1}{2} \sigma_t^{i\varrho}(x) \sigma_t^{j\varrho}(x) \partial_{ij} \phi^l(x) + \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{i\varrho}(x) a_t^{\bar{l}\varrho}(x) \partial_i \phi^{\bar{l}}(x) \\ &\quad + \int_{D^k} \rho_t^{\bar{l}}(x, z) \left(\phi^{\bar{l}}(x + H_t(x, z)) - \phi^{\bar{l}}(x) \right) \pi(dz) \\ &\quad + \int_{D^k} \left(\phi^l(x + H_t(x, z)) - \phi^l(x) - \mathbf{1}_{\{1,2\}}(\alpha) \partial_i \phi^l(x) H_t^i(x, z) \right) \pi(dz) \end{aligned}$$

$$\mathcal{N}_t^{l\varrho} \phi^l(x) := \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{i\varrho}(x) \partial_i \phi^l(x) + v_t^{\bar{l}\varrho}(x) \phi^{\bar{l}}(x),$$

$$I_{t,z}^l \phi^l(x) := (I_{d_2} + \rho_t^{\bar{l}}(x, z)) \phi^{\bar{l}}(x + H_t(x, z)) - \phi^l(x),$$

and where

$$\begin{aligned}
\hat{b}_t^i(x) &:= \mathbf{1}_{[1,2]}(\alpha)b_t^i(x) + \mathbf{1}_{[2]}(\alpha)\sigma_t^{j\varrho}(x)\partial_j\sigma_t^{i\varrho}(x) \\
&\quad + \int_D \left(\mathbf{1}_{(1,2]}(\alpha)H_t^i(x, z) - H_t^i(\tilde{H}_t^{-1}(x, z), z) \right) \pi(dz), \\
\hat{c}_t^{\bar{l}}(x) &:= c_t^{\bar{l}}(x) + \mathbf{1}_{[2]}(\alpha)\sigma_t^{j\varrho}(x)\partial_j v_t^{\bar{l}\varrho}(x) + \int_D \left(\rho_t^{\bar{l}}(x, z) - \rho_t^{\bar{l}}(\tilde{H}_t^{-1}(x, z), z) \right) \pi(dz), \\
\hat{f}_t^l(x) &:= f_t^l(x) + \mathbf{1}_{[2]}(\alpha)\sigma_t^{j\varrho}(x)\partial_j g_t^l(x) + \int_D \left(h_t^l(x, z) - h_t^l(\tilde{H}_t^{-1}(x, z), z) \right) \pi(dz).
\end{aligned}$$

We associate with (3.15) the stochastic flow $X_t = X_t(x) = X_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, given by (3.8). Let $\Gamma_t(x) = \Gamma_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, be the solution of the linear SDE given by

$$\begin{aligned}
d\Gamma_t(x) &= (c_t(X_t(x))\Gamma_t(x) + f_t(X_t(x)))dt + (v_t^{\varrho}(X_t(x))\Gamma_t(x) + g_t^{\varrho}(X_t(x)))dw_t^{\varrho} \\
&\quad + \int_Z \rho_t(\tilde{H}_t^{-1}(X_{t-}(x), z), z)\Gamma_{t-}(x)[\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)] \\
&\quad + \int_Z h_t(\tilde{H}_t^{-1}(X_{t-}(x), z), z)[\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)], \quad \tau < t \leq T, \\
\Gamma_t(x) &= 0, \quad t \leq \tau.
\end{aligned} \tag{3.16}$$

Let $\Psi_t(x) = \Psi_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, be the unique solution of the linear SDE given by

$$\begin{aligned}
d\Psi_t(x) &= c_t(X_t(x))\Psi_t(x)dt + v_t^{\varrho}(X_t(x))\Phi_t(x)dw_t^{\varrho} \\
&\quad + \int_Z \rho_t(\tilde{H}_t^{-1}(X_{t-}(x), z), z)\Psi_{t-}(x)[\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)], \quad \tau < t \leq T, \\
\Psi_t(x) &= I_{d_2}, \quad t \leq \tau.
\end{aligned}$$

In the following lemma, we obtain p -th moment estimates of the weighted Hölder norms of Γ and Ψ .

Lemma 3.2. *Let Assumptions 3.2($\bar{\beta}$) and 3.3($\bar{\beta}$) hold for some $\bar{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$. For each stopping time $\tau \leq T$ and $\beta' \in [0, \bar{\beta} \wedge \tilde{\beta}]$, there exists a $D([0, T], C_{loc}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$ -modification of $\Gamma(\tau)$ and $\Psi(\tau)$, also denoted by $\bar{\Gamma}(\tau)$ and $\bar{\Psi}(\tau)$, respectively. Moreover, for each $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d_1, d_2, p, N_0, T, \beta', \eta, \epsilon, \theta)$ such that*

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(\theta+\epsilon)} \Gamma_t(\tau)|_{\beta'}^p \right] + \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \Psi_t(\tau)|_{\beta'}^p \right] \leq N. \tag{3.17}$$

Proof. Let $\tau \leq T$ be a fixed stopping time and $\beta := \bar{\beta} \wedge \tilde{\beta}$. Estimating (3.16) directly and using the Burkholder-Davis-Gundy inequality, Lemma 4.1, the multiplicative decomposition

$$h_t(x, \tilde{H}_t^{-1}(X_{t-}(x), z), z) = r_1^{\theta}(X_{t-}(x)) \frac{r_1^{\theta}(\tilde{H}_t^{-1}(X_{t-}(x), z))}{r_1^{\theta}(X_t(x))} \frac{h_t(\tilde{H}_t^{-1}(X_{t-}(x), z), z)}{r_1^{\theta}(\tilde{H}_t^{-1}(X_{t-}(x), z))},$$

Hölder's inequality, Lemma 4.10 (1), Lemma 3.2 in [LM14b], and Gronwall's inequality, we get that for all x and y ,

$$\mathbf{E} \left[\sup_{t \leq T} |\Gamma_t(x)|^p \right] \leq N r_1^{-\theta p}(x)$$

and

$$\mathbf{E} \left[\sup_{t \leq T} |\Gamma_t(x) - \Gamma_t(y)|^p \right] \leq N(r_1^{-p\theta}(x) \vee r_1^{-p\theta}(y)) |x - y|^{(\beta' \wedge 1)p},$$

where $N = N(d_1, p, N_0, T, \eta, \theta)$ is a positive constant. Now, assume that $[\beta]^- \geq 1$. As in the proof of Theorem 3.4 in [Kun04], it follows that $\mathfrak{U}_t = \nabla \Gamma_t(\tau, x)$ solves

$$\begin{aligned} d\mathfrak{U}_t &= (v_t^\rho(X_t)\mathfrak{U}_t + \nabla v_t^\rho(X_t)\nabla X_t\Gamma_t + \nabla g_t^\rho(X_t)\nabla X_t)dw_t^\rho \\ &\quad + \int_Z \rho_t(\tilde{H}_t^{-1}(X_{t-}, z), z)\mathfrak{U}_{t-}[\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)] \\ &\quad + \int_Z \nabla \rho_t(\tilde{H}_t^{-1}(X_{t-}, z), z)\nabla[\tilde{H}_t^{-1}(X_{t-})]\Gamma_{t-}[\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)] \\ &\quad + \int_Z \nabla h_t(x, \tilde{H}_t^{-1}(X_t, z), z)\nabla[\tilde{H}_t^{-1}(X_{t-})][\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)] \\ &\quad + (c_t(X_t)\mathfrak{U}_t + \nabla c_t(X_t)\nabla X_t\Gamma_t + \nabla f_t(X_t)\nabla X_t)dt, \quad \tau < t \leq T, \\ \mathfrak{U}_t &= 0, \quad t \leq \tau. \end{aligned}$$

Recall that by Lemma 4.6, a function $\phi : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^n$, $n \geq 1$ satisfies $|r^{-\theta}\phi|_\beta < \infty$ if and only if $|r^{-\theta}\phi|_0, \dots, |r^{-\theta}\partial^\gamma\phi|_0$, $|\gamma| \leq [\beta]^-$, and $[r^{-\theta}\partial^\gamma\phi]_{|\beta|+}$ are finite. Estimating as above and using Proposition 3.4 in [LM14b], we obtain that for each $p \geq 2$, there is a constant $N = N(d_1, d_2, p, N_0, T, \theta)$ such that for all x and y ,

$$\mathbf{E} \left[\sup_{t \leq T} |\nabla \Gamma_t(x)|^p \right] \leq r_1^{-p\theta}(x)N$$

and

$$\mathbf{E} \left[\sup_{t \leq T} |\nabla \Gamma_t(x) - \nabla \Gamma_t(y)|^p \right] \leq N(r_1^{-p\theta}(x) \vee r_1^{-p\theta}(y)) |x - y|^{((\beta-1) \wedge 1)p}.$$

Using induction, we get that for each $p \geq 2$ and all multi-indices γ with $0 \leq |\gamma| \leq [\beta]^-$ and all x ,

$$\mathbf{E} \sup_{t \leq T} [|\partial^\gamma \Gamma_t(x)|^p] \leq r_1^{-p\theta}(x)N,$$

and for all multi-indices γ with $|\gamma| = [\beta]^-$ and all x, y ,

$$\mathbf{E} \left[\sup_{t \leq T} |\partial^\gamma \Gamma_t(x) - \partial^\gamma \Gamma_t(y)|^p \right] \leq N(r_1^{-p\theta}(x) \vee r_1^{-p\theta}(y)) |x - y|^{(\beta - [\beta]^-)p},$$

for a constant $N = N(d_1, d_2, p, N_0, T, \beta, \eta, \theta)$. It is also clear that for each $p \geq 2$ and all multi-indices γ with $0 \leq |\gamma| \leq [\beta]^-$ and all x ,

$$\mathbf{E} \left[\sup_{t \leq T} |\partial^\gamma \Psi_t(x)|^p \right] \leq N,$$

and for all multi-indices γ with $|\gamma| = [\beta]^-$ and all x, y ,

$$\mathbf{E} \left[\sup_{t \leq T} |\partial^\gamma \Psi_t(x) - \partial^\gamma \Psi_t(y)|^p \right] \leq N |x - y|^{(\beta - [\beta]^-)p}.$$

We obtain the existence of a $D([0, T], C_{loc}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$ -modification of $\Gamma(\tau)$ and $\Psi(\tau)$ using estimate (3.17) and Corollary 5.4 in [LM14b]. This completes the proof. \square

Let $\tilde{\Phi}_t(x) = \tilde{\Phi}_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, be the solution of the linear SDE given by

$$\begin{aligned} d\tilde{\Phi}_t(x) &= \left(c_t(X_t(x))\tilde{\Phi}_t(x) + f_t(X_t(x)) \right) dt + \left(v_t^o(X_t(x))\tilde{\Phi}_t(x) + g_t^o(X_t(x)) \right) dw_t^o \\ &\quad + \int_Z \rho_t(\tilde{H}_t^{-1}(X_{t-}(x), z), z) \tilde{\Phi}_{t-}(x, y) [\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)] \\ &\quad + \int_Z h_t(\tilde{H}_t^{-1}(X_{t-}(x), z), z) [\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)], \quad \tau < t \leq T, \\ \tilde{\Phi}_t(x) &= \varphi(x), \quad t \leq \tau. \end{aligned}$$

The following is a simple corollary of Lemma 3.2.

Corollary 3.3. *Let Assumptions 3.2($\bar{\beta}$) and 3.3($\tilde{\beta}$) hold for some $\bar{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$. For each stopping time $\tau \leq T$ and $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable random field φ such that for some $\beta' \in [0, \bar{\beta} \wedge \tilde{\beta}]$, \mathbf{P} -a.s. $\varphi \in C_{loc}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$, there is a $D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^{d_1}, \mathbf{R}^{d_2}))$ -modification of $\tilde{\Phi}(\tau)$, also denoted by $\tilde{\Phi}(\tau)$, and \mathbf{P} -a.s. for all $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$,*

$$\tilde{\Phi}_t(\tau, x) = \Psi_t(x)\varphi(x) + \Gamma_t(x).$$

Moreover, if for some $\theta' \geq 0$ and $\beta' \in [0, \bar{\beta} \wedge \tilde{\beta}]$, \mathbf{P} -a.s. $r_1^{-\theta'} \varphi \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$, then for each $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d_1, d_2, p, N_0, T, \theta, \theta', \beta', \epsilon)$ such that

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(\theta \vee \theta') - \epsilon} \tilde{\Phi}_t(\tau)|_{\beta'}^p \middle| \mathcal{F}_\tau \right] \leq N(|r_1^{-\theta'} \varphi|_{\beta'}^p + 1). \quad (3.18)$$

Now we are ready to state our main result concerning fully-degenerate SDEs and their connection with linear transformations of inverse flows of jump SDEs.

Proposition 3.4. *Let Assumptions 3.2($\bar{\beta}$) and 3.3($\tilde{\beta}$) hold for some $\bar{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$. For each stopping time $\tau \leq T$ and $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable random field φ such that for some $\beta' \in (\alpha, \bar{\beta} \wedge \tilde{\beta})$ and $\theta' \geq 0$, \mathbf{P} -a.s. $r_1^{-\theta'} \varphi \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$, we have that \mathbf{P} -a.s. $\tilde{\Phi}(\tau, X^{-1}(\tau)) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$ and $v_t(x) = v_t(\tau, x) = \tilde{\Phi}_t(\tau, X_t^{-1}(\tau, x))$ solves (3.15). Moreover, for each $\epsilon > 0$ and $p \geq 2$,*

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(\theta \vee \theta') - \epsilon} v_t(\tau)|_{\beta'}^p \middle| \mathcal{F}_\tau \right] \leq N(|r_1^{-\theta'} \varphi|_{\beta'}^p + 1), \quad (3.19)$$

for a constant $N = N(d_1, d_2, p, N_0, T, \beta', \eta, \epsilon, \theta, \theta')$.

Proof. Fix a stopping time $\tau \leq T$ and random field φ such that for some $\beta' \in (\alpha, \bar{\beta} \wedge \tilde{\beta})$ and $\theta' \geq 0$, \mathbf{P} -a.s. $r_1^{-\theta'} \varphi \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$. By virtue of Corollary 3.3 and Theorem 2.1 in [LM14b], \mathbf{P} -a.s.

$$\tilde{\Phi}(\tau, X^{-1}(\tau)) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^{d_1}, \mathbf{R}^{d_2})).$$

Then using the Ito-Wenzell formula (Proposition 4.16) and following a simple calculation, we obtain that $v_t(\tau, x) := \tilde{\Phi}_t(\tau, X_t^{-1}(\tau, x))$ solves (3.15). By Theorem 2.1 in [LM14b] and Corollary 3.3, for each $\epsilon > 0$ and $p \geq 2$, there exists a constant $N = N(d_1, p, N_0, T, \beta', \eta, \epsilon)$ such that

$$\mathbf{E}[\sup_{t \leq T} |r_1^{-(1+\epsilon)} X_t^{-1}(\tau)|_{\beta'}^p] + \mathbf{E}[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{-1}(\tau)|_{\beta'-1}^p] \leq N. \quad (3.20)$$

Therefore applying Lemma 4.9 and Hölder's inequality and using the estimates (3.20) and (3.18), we obtain (3.19), which completes the proof. \square

3.4 Adding uncorrelated part (Proof of Theorem 2.2)

Proof of Theorem 2.2. Fix a stopping time $\tau \leq T$ and random field φ such that for some $\beta' \in (\alpha, \bar{\beta} \wedge \tilde{\beta})$ and $\theta' \geq 0$, \mathbf{P} -a.s. $r_1^{-\theta'} \varphi \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$. Consider the system of SIDs given by

$$\begin{aligned} d\tilde{v}_t^l &= \left((\mathcal{L}_t^{1;l} + \mathcal{L}_t^{2;l}) \tilde{v}_t + \mathbf{1}_{[1,2]}(\alpha) \hat{b}_t^i \partial_i u_t^l + \hat{c}_t^{\bar{l}} u_t^{\bar{l}}(x) + \hat{f}_t^l \right) dt + \left(\mathcal{N}_t^{1;l\varrho} \tilde{v}_t + g_t^{l\varrho} \right) dW_t^{1;\varrho} \\ &\quad + \mathcal{N}_t^{2;l\varrho} \tilde{v}_t dW_t^{2;\varrho} + \int_{Z^1} \left(\mathcal{I}_{t,z}^{1;l} \tilde{v}_{t-} + h_t^l(z) \right) [\mathbf{1}_{D^1}(z) q^1(dt, dz) + \mathbf{1}_{E^1} p^1(dt, dz)] \\ &\quad + \int_{Z^2} \mathcal{I}_{t,z}^{2;l} \tilde{v}_{t-} [\mathbf{1}_{D^2}(z) q^2(dt, dz) + \mathbf{1}_{E^2}(z) p^2(dt, dz)] \quad \tau < t \leq T, \\ \tilde{v}_t^l &= \varphi^l, \quad t \leq \tau, \quad l \in \{1, \dots, d_2\}, \end{aligned}$$

where for $\phi \in C_c^\infty(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ and $l \in \{1, \dots, d_2\}$,

$$\begin{aligned} \mathcal{N}_t^{2;l\varrho} \phi(x) &:= \mathbf{1}_{[2]}(\alpha) \sigma_t^{2;l\varrho}(x) \partial_i \phi^l(x) + v_t^{2;l\bar{l}\varrho}(x) \phi^{\bar{l}}(x), \quad \varrho \geq 1, \\ \mathcal{I}_{t,z}^{2;l} \phi(x) &:= (I_{d_2}^{\bar{l}} + \rho_t^{2;l\bar{l}}(x, z)) \phi^{\bar{l}}(x + H_t^2(x, z)) - \phi^l(x). \end{aligned}$$

By Proposition 3.4, \mathbf{P} -a.s. $\Phi(\tau, X^{-1}(\tau)) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$ and $\tilde{v}_t(\tau, x) = \Phi_t(\tau, X_t^{-1}(\tau, x))$ solves (3.15). We write $v_t(x) = v_t(\tau, x)$. Moreover, for each $\epsilon > 0$ and $p \geq 2$,

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(\theta \vee \theta') - \epsilon} \tilde{v}_t(\tau)|_{\beta'}^p \middle| \mathcal{F}_\tau \right] \leq N(|r_1^{-\theta'} \varphi|_{\beta'}^p + 1), \quad (3.21)$$

where $N = N(d_1, d_2, p, N_0, T, \beta', \eta^1, \eta^2, \epsilon, \theta, \theta')$ is a positive constant. Without loss of generality we will assume that for all ω and t , $|r_1^{-\theta'} \varphi|_{\beta'} \leq N$, since we can always multiply the equation by indicator function. For each $n \in \mathbf{N} \cup \{0\}$, let $C_{loc}^n(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ be the separable

Fréchet space of n -times continuously differentiable functions $f : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_2}$ endowed with the countable set of semi-norms given by

$$|f|_{n,int} = \sum_{0 \leq |\gamma| \leq n} \sup_{|x| \leq k} |\partial^\gamma f(x)|, \quad k \in \mathbf{N}.$$

Owing to Lemma 4.2, there is a family of measures $E_\omega^t(dU)$, $(\omega, t) \in \Omega \times [0, T]$ on $D([0, T]; C_{loc}^{[\beta]^-}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$, corresponding to $\mathfrak{A} = \tilde{v}$ such that for all bounded $G : \Omega \times [0, T] \times [0, T] \times D([0, T]; C_{loc}^{[\beta]^-}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})) \rightarrow \mathbf{R}^{d_2}$ that are $\mathcal{O}_{\mathcal{T}} \times \mathcal{B}([0, T]) \times \mathcal{B}(D([0, T]; C_{loc}^{[\beta]^-}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})))$ measurable, \mathbf{P} -a.s. for all t , we have

$$E^t[G_t(t, \tilde{v})] = \int_{D([0, T]; C_{loc}^{[\beta]^-}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))} G_t(t, U) E^t(dU) = \mathbf{E}[G_t(t, \tilde{v}) | \mathcal{F}_t],$$

where the right-hand-side is the càdlàg modification of the conditional expectation. Set

$$\hat{u}_t(x) = \hat{u}_t(\tau, x) = E^t[\tilde{v}_t(\tau, x)] = \int_{D([0, T]; C_{loc}^{[\beta]^-}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))} U_t(x) E^t(dU).$$

Let $\lambda = (\theta \vee \theta') + \epsilon$. We claim that for all multi-indices γ with $|\gamma| \leq [\beta]^-$, \mathbf{P} -a.s. for all t and x ,

$$\partial^\gamma[r_1^{-\lambda}(x)\hat{u}_t(x)] = \int_{D([0, T]; C_{loc}^{[\beta]^-}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))} \partial^\gamma[r_1^{-\lambda}(x)U_t(x)] E^t(dU) = E^t[\partial^\gamma[r_1^{-\lambda}(x)\tilde{v}_t(x)]].$$

Indeed, since

$$M_t = E^t \left[\sup_{s \leq T} |\partial^\gamma[r_1^{-\lambda}\tilde{v}_s]|_0 \right], \quad t \in [0, T],$$

is a (\mathbf{F}, \mathbf{P}) martingale, we have

$$\mathbf{E} \left[\sup_{t \leq T} |M_t|^2 \right] \leq 4\mathbf{E} \left[|M_T|^2 \right] \leq 4\mathbf{E} \left[\sup_{t \leq T} |\partial^\gamma[r_1^{-\lambda}\tilde{v}_t]|_0^2 \right] < \infty, \quad (3.22)$$

and hence \mathbf{P} -a.s. for all t ,

$$\int_{D([0, T]; C_{loc}^{[\beta]^-}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))} \sup_{s \leq T, x \in \mathbf{R}^{d_1}} |\partial^\gamma[r_1^{-\lambda}(x)U_s(x)]| E^t(dU) = E^t \left[\sup_{t \leq T} |\partial^\gamma[r_1^{-\lambda}\tilde{v}_t]|_0 \right] < \infty.$$

Similarly, since $\mathbf{E} \left[\sup_{t \leq T} |r_1^{-\lambda}\tilde{v}_t|_{\beta'}^2 \right] < \infty$, \mathbf{P} -a.s. for each x and y ,

$$\begin{aligned} \frac{|\partial^\gamma[r_1^{-\lambda}(x)\hat{u}_t(x)] - \partial^\gamma[r_1^{-\lambda}(y)\hat{u}_t(y)]|}{|x - y|^{[\beta']^+}} &\leq E^t \left[\frac{|\partial^\gamma[r_1^{-\lambda}(x)\tilde{v}_t(x)] - \partial^\gamma[r_1^{-\lambda}(y)\tilde{v}_t(y)]|}{|x - y|^{[\beta']^+}} \right] \\ &\leq E^t[|r_1^{-\lambda}\tilde{v}_t|_{\beta'}], \end{aligned}$$

and hence, \mathbf{P} -a.s.

$$\sup_{t \leq T} |r_1^{-\lambda}\hat{u}_t|_{\beta'} \leq \sup_{t \leq T} E^t \left[\sup_{t \leq T} |r_1^{-\lambda}\tilde{v}_t|_{\beta'} \right] < \infty.$$

Thus, \mathbf{P} -a.s. $r_1^{-\lambda}(\cdot)\hat{u}(\tau) \in D([0, T]; C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$ and (2.4) follows from (3.21) (see the argument (3.22)). For each $l \in \{1, \dots, d_2\}$, let

$$\begin{aligned} \mathcal{A}_t^l(x) &= \varphi^l(x) + \int_{] \tau, \tau \vee t]} \left((\mathcal{L}_s^{1;l} + \mathcal{L}_s^{2;l})\hat{u}_s(x) + \mathbf{1}_{[1,2]}(\alpha)\hat{b}_s^i(x)\partial_i\hat{u}_s^l(x) + \hat{c}_s^{\bar{l}}(x)\hat{u}_s^{\bar{l}}(x) + \hat{f}_s^l(x) \right) ds \\ &\quad + \int_{] \tau, \tau \vee t]} \left(\mathcal{N}_s^{1;l\varrho}\hat{u}_s(x) + g_s^{l\varrho}(x) \right) dw_s^{1;\varrho} \\ &\quad + \int_{] \tau, \tau \vee t]} \int_{Z^1} \left(\mathcal{I}_{s,z}^{1;l}\hat{u}_{s-}(x) + h_s^l(x, z) \right) [\mathbf{1}_{D^1}(z)q^1(ds, dz) + \mathbf{1}_{E^1}(z)p^1(ds, dz)]. \end{aligned}$$

By Theorem 12.21 in [Jac79], the representation property holds for (\mathbf{F}, \mathbf{P}) , and hence every bounded (\mathbf{F}, \mathbf{P}) -martingale issuing from zero can be represented as

$$M_t = \int_{]0,t]} o_s^{\varrho} dw_s^{1;\varrho} + \int_{]0,t]} \int_{Z^1} e_s(z)q^1(ds, dz), \quad t \in [0, T],$$

where

$$\mathbf{E} \int_{]0,T]} |o_s|^2 ds + \mathbf{E} \int_{]0,T]} \int_{Z^1} |e_s(z)|^2 \pi^1(dz) ds < \infty.$$

Then for an arbitrary \mathbf{F} -stopping time $\bar{\tau} \leq T$ and bounded (\mathbf{F}, \mathbf{P}) -martingale, applying Itô's product rule and taking the expectation, we obtain

$$\mathbf{E} \tilde{v}_{\bar{\tau}}(\tau, x) \bar{M}_{\bar{\tau}} = \mathbf{E} \mathcal{A}_{\bar{\tau}}(x) \bar{M}_{\bar{\tau}}.$$

Since the optional projection is unique, \mathbf{P} -a.s. for all t and x , $\hat{u}_t(x) = \mathcal{A}_t(x)$. This completes the proof. \square

3.5 Interlacing a sequence of large jumps (Proof of Theorem 2.5)

Proof of Theorem 2.5. Fix a stopping time $\tau \leq T$ and random field φ such that for some $\beta' \in (\alpha, \bar{\beta} \wedge \tilde{\beta})$ and $\theta' \geq 0$, \mathbf{P} -a.s. $r_1^{-\theta'} \varphi \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$. For any $\delta > 0$, we can rewrite (1.1) as

$$\begin{aligned} du_t^l &= \left((\bar{\mathcal{L}}_t^{1;l} + \mathcal{L}_t^{2;l})u_t + \mathbf{1}_{[1,2]}(\alpha)\bar{b}_t^i\partial_i u_t^l + \bar{c}_t^{\bar{l}}u_t^{\bar{l}} + f_t^l \right) dt + \left(\mathcal{N}_t^{1;l\varrho}u_t + g_t^{l\varrho} \right) dw_t^{1;\varrho} \\ &\quad + \int_{Z^1} \left(\bar{\mathcal{I}}_{t,z}^{1;l}u_{t-} + \bar{h}_t^l(z) \right) [\mathbf{1}_{D^1}(z)q^1(dt, dz) + \mathbf{1}_{E^1}(z)p^1(dt, dz)] \\ &\quad + \int_{Z^1} \left(\mathbf{1}_{(D^1 \cup E^1) \cap \{K_t^1 > \delta\}}(z) + \mathbf{1}_{V^1}(z) \right) \left(\mathcal{I}_{t,z}^{1;l}u_{t-} + h_t^l(z) \right) p^1(dt, dz), \quad \tau < t \leq T, \\ u_t^l &= \varphi^l, \quad t \leq \tau, \quad l \in \{1, \dots, d_2\}, \end{aligned} \tag{3.23}$$

where for $\phi \in C_c^\infty(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ and $l \in \{1, \dots, d_2\}$,

$$\begin{aligned} \bar{\mathcal{L}}_t^{1;l}\phi(x) &:= \mathbf{1}_{\{2\}}(\alpha) \frac{1}{2} \sigma_t^{1;i\varrho}(x) \sigma_t^{1;j\varrho}(x) \partial_{ij} \phi^l(x) + \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{k;i\varrho}(x) v_t^{1;\bar{l}\varrho}(x) \partial_i \phi^{\bar{l}}(x) \\ &\quad + \int_{D^1} \bar{\rho}_t^{1;\bar{l}}(x, z) \left(\phi^{\bar{l}}(x + \bar{H}_t^1(x, z)) - \phi^{\bar{l}}(x) \right) \pi^1(dz) \\ &\quad + \int_{D^1} \left(\phi^l(x + \bar{H}_t^1(x, z)) - \phi^l(x) - \mathbf{1}_{\{1,2\}}(\alpha) \bar{H}_t^{1;i}(x, z) \partial_i \phi^l(x) \right) \pi^1(dz), \end{aligned}$$

$$\begin{aligned}
\bar{I}_{t,z}^1 \phi^l(x) &= (I_{d_2}^{\bar{l}} + \mathbf{1}_{\{K_t^1 \leq \delta\}}(z) \rho_t^{1;\bar{l}}(x, z)) \phi^{\bar{l}}(x + \mathbf{1}_{\{K_t^1 \leq \delta\}}(z) H_t^1(x, z)) - \phi^l(x), \\
\bar{H}^1 &:= \mathbf{1}_{\{K_t^1 \leq \delta\}} H^1, \quad \bar{\rho}^1 := \mathbf{1}_{\{K_t^1 \leq \delta\}} \rho^1, \quad \bar{h} := \mathbf{1}_{\{K_t^1 \leq \delta\}} h, \\
\bar{b}_t^i(x) &:= b_t^i(x) - \int_{D^1 \cap \{K_t^1 > \delta\}} \mathbf{1}_{(1,2]}(\alpha) H_t^{1;i}(x, z) \pi^1(dz), \\
\bar{c}_t^{\bar{l}}(x) &:= c_t^{\bar{l}}(x) - \int_{D^1 \cap \{K_t^1 > \delta\}} \rho_t^{1;\bar{l}}(x, z) \pi^1(dz).
\end{aligned}$$

For an arbitrary stopping time $\tau' \leq T$ and $\mathcal{F}_{\tau'} \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable random field $\varphi^{\tau'} : \Omega \times \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_2}$ satisfying for some $\theta(\tau') > 0$, \mathbf{P} -a.s. $r_1^{-\theta(\tau')} \varphi^{\tau'} \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$, consider the system of SDEs on $[0, T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned}
dv_t^l &= \left((\bar{\mathcal{L}}_t^{1;l} + \mathcal{L}_t^{2;l}) v_t + \mathbf{1}_{[1,2]}(\alpha) \bar{b}_t^i \partial_i v_t^l + \bar{c}_t^{\bar{l}} v_t^{\bar{l}} + f_t^l \right) dt + \left(\mathcal{N}_t^{1;l\varrho} v_t + g_t^{l\varrho} \right) dw_t^{1;\varrho} \\
&\quad + \int_{Z^1} \left(\bar{I}_{t,z}^{1;l} u_{t-} + \bar{h}_t^l(z) \right) [1_{D^1}(z) q^1(dt, dz) + 1_{E^1}(z) p^1(dt, dz)], \quad \tau' < t \leq T, \\
v_t^l &= \varphi^{\tau';l}, \quad t \leq \tau', \quad l \in \{1, \dots, d_2\}.
\end{aligned} \tag{3.24}$$

Set $\bar{H}^2 = H^2$ and $\bar{\rho}^2 = \rho^2$. In order to invoke Theorem 2.2 and obtain a unique solution $v_t = v_t(\tau', x) = v_t(\tau', \varphi^{\tau'}, x)$ of (3.24), we will show that for all ω and t ,

$$|r_1^{-1} \tilde{b}_t|_0 + |\nabla \tilde{b}_t|_{\bar{\beta}-1} + |\tilde{c}_t|_{\bar{\beta}} + |r^{-\theta} \tilde{f}|_{\bar{\beta}} \leq N_0, \tag{3.25}$$

where

$$\begin{aligned}
\tilde{b}_t^i(x) &:= \mathbf{1}_{[1,2]}(\alpha) \bar{b}_t^i(x) - \sum_{k=1}^2 \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{k;j\varrho}(x) \partial_j \sigma_t^{k;i\varrho}(x) \\
&\quad - \sum_{k=1}^2 \int_{D^k} \left(\mathbf{1}_{(1,2]}(\alpha) \bar{H}_t^{k;i}(x, z) - \bar{H}_t^{k;i}(\tilde{H}_t^{k;-1}(x, z), z) \right) \pi^k(dz), \\
\tilde{c}_t^{\bar{l}}(x) &:= \bar{c}_t^{\bar{l}}(x) - \sum_{k=1}^2 \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{k;i\varrho}(x) \partial_i v_t^{k;\bar{l}\varrho}(x) \\
&\quad - \sum_{k=1}^2 \int_{D^k} \left(\bar{\rho}_t^{k;\bar{l}}(x, z) - \bar{\rho}_t^{k;\bar{l}}(\tilde{H}_t^{k;-1}(x, z), z) \right) \pi^k(dz), \\
\tilde{f}_t^l(x) &:= f_t^l(x) - \sigma_t^{1;j\varrho}(x) \partial_j g_t^l(x) - \int_{D^1} \left(\bar{h}_t^l(x, z) - \bar{h}_t^l(\tilde{H}_t^{1;-1}(x, z), z) \right) \pi^1(dz).
\end{aligned}$$

Owing to Assumption 2.3($\bar{\beta}, \delta^1, \delta^2, \mu^1, \mu^2$), we easily deduce that there is a constant $N = N(d_1, N_0, \bar{\beta})$ such that for each $k \in \{1, 2\}$ and all ω and t ,

$$|\sigma_t^{k;j\varrho} \partial_j \sigma_t^{k;\varrho}|_{\bar{\beta}} + |\sigma_t^{k;j\varrho} \partial_j \sigma_t^{k;\varrho}(x)|_{\bar{\beta}} + |\sigma_t^{1;j\varrho} \partial_j g_t^{\varrho}|_{\bar{\beta}} \leq N, \quad \text{if } \alpha = 2.$$

Since $|\nabla \bar{H}_t^1|_0 \leq \delta$, for any fixed $\eta^1 < 1$, for all $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^{d_1} \times (D^1 \cup E^1) : |\nabla \bar{H}_t^1(\omega, x, z)| > \eta^1\}$,

$$\left| \left(I_{d_1} + \nabla H_t^1(\omega, x, z) \right)^{-1} \right| \leq \frac{1}{1 - \delta}.$$

Appealing to Assumption 2.3($\bar{\beta}, \delta^1, \delta^2, \mu^1, \mu^2$) and applying Lemma 4.10, we obtain that there is a constant $N = N(d_1, d_2, N_0)$ such that for each $k \in \{1, 2\}$ and all ω, t , and z ,

$$\begin{aligned} |\bar{H}_t^{k;i}(z) - \bar{H}_t^{k;i}(\tilde{H}_t^{k;-1}(z), z)|_{\bar{\beta}} &\leq N(K_t^k(z) + \bar{K}_t^k(z))^2 + N\mathbf{1}_{(0,1]}(\{\bar{\beta}\}^+ + \delta^k) \tilde{K}_t^k(z) K_t^k(z)^{\delta^k} \\ &\quad + N\mathbf{1}_{(1,2]}(\{\bar{\beta}\}^+ + \delta^k) (\tilde{K}_t^k(z) K_t^k(z)^{\delta^k} + \bar{K}_t^k(z)^2), \\ |\bar{\rho}_t^k(z) - \bar{\rho}_t^k(\tilde{H}_t^{k;-1}(z), z)|_{\bar{\beta}} &\leq Nl_t^k(z)(K_t^k(z) + \bar{K}_t^k(z)) + N\mathbf{1}_{(0,1]}(\{\bar{\beta}\}^+ + \mu^k) \tilde{l}_t^k(z) K_t^k(z)^{\mu^k} \\ &\quad + N\mathbf{1}_{(1,2]}(\{\bar{\beta}\}^+ + \mu^k) (\tilde{l}_t^k(z) K_t^k(z)^{\mu^k} + l_t^k(z) \bar{K}_t^k(z)), \end{aligned}$$

and

$$\begin{aligned} |r_1^{-\theta} \bar{h}_t(z) - r_1^{-\theta} \bar{h}_t(\tilde{H}_t^{1;-1}(z), z)|_{\bar{\beta}} &\leq Nl_t^1(z)(K_t^1(z) + \bar{K}_t^1(z)) + N\mathbf{1}_{(0,1]}(\{\bar{\beta}\}^+ + \mu^1) \tilde{l}_t^1(z) K_t^1(z)^{\mu^1} \\ &\quad + N\mathbf{1}_{(1,2]}(\{\bar{\beta}\}^+ + \mu^1) (\tilde{l}_t^1(z) K_t^1(z)^{\mu^1} + l_t^1(z) \bar{K}_t^1(z)). \end{aligned}$$

Moreover, using Lemma 4.10, we find that there is a constant $N = N(d_1, d_2, N_0)$ such that for each $k \in \{1, 2\}$, and all ω, t , and z ,

$$|r_1^{-1} \bar{H}_t^k(\tilde{H}_t^{k;-1}(z), z)|_0 \leq |r_1^{-1} H^k|_0, \quad |\nabla[\bar{H}_t^{k;i}(\tilde{H}_t^{k;-1}(z), z)]|_{\bar{\beta}} \leq |\nabla H^k|_{\bar{\beta}-1}.$$

Combining the above estimates and using Hölder's inequality and the integrability properties of $l_t^k(z)$ and $K_t^k(z)$, we obtain (3.25). Therefore, by Theorem 2.2, for each stopping time $\tau' \leq T$ and and $\mathcal{F}_{\tau'} \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable random field $\varphi^{\tau'}$ satisfying for some $\theta(\tau') > 0$, \mathbf{P} -a.s. $r_1^{-\theta(\tau')} \varphi^{\tau'} \in C^{\beta'}(\mathbf{R}^{d_1}, \mathbf{R}^{d_2})$, there exists a unique solution $v_t(x) = v_t(\tau', \varphi^{\tau'}, x)$ of (3.24) such that

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-\theta(\tau') \vee \theta - \epsilon} v_t(\tau')|_{\beta'}^p \middle| \mathcal{F}_{\tau'} \right] \leq N(|r_1^{-\theta(\tau')} \varphi^{\tau'}|_{\beta'}^p + 1), \quad (3.26)$$

where $N = N(d_1, d_2, p, N_0, T, \beta', \eta^1, \eta^2, \epsilon, \theta, \theta(\tau'))$ is a positive constant. Let

$$A_t = \int_{[0,t]} \int_{Z^1} (\mathbf{1}_{(D^1 \cup E^1) \cap \{K_s^1 > \eta^1\}}(z) + \mathbf{1}_{V^1}(z)) p^1(ds, dz), \quad t \leq T.$$

Define a sequence of stopping times $(\tau_n)_{n \geq 0}$ recursively by $\tau_1 = \tau$ and

$$\tau_{n+1} = \inf(t > \tau_n : \Delta A_t \neq 0) \wedge T.$$

We obtain the existence of a unique solution $u = u(\tau)$ of (3.23) in $\mathfrak{G}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ by interlacing solutions of (3.24) along the sequence of stopping times (τ_n) . For $(\omega, t) \in [[0, \tau_1))$, we set $u_t(\tau, x) = v_t(\tau, \varphi, x)$ and note that

$$\mathbf{E} \left[\sup_{t \leq \tau_1} |r_1^{-\theta' \vee \theta - \epsilon} u_t(\tau)|_{\beta'}^p \middle| \mathcal{F}_{\tau} \right] \leq N(|r_1^{-\theta'} \varphi|_{\beta'}^p + 1).$$

For each ω and x , we set

$$u_{\tau_1}(x) = u_{\tau_1-}(x) + \int_{Z^1} (\mathbf{1}_{(D^1 \cup E^1) \cap \{K^1 > \eta^1\}}(t, z) + \mathbf{1}_{V^1}(z)) (I_{t,z}^1 u_{\tau_1-}(x) + h_{\tau_1}^1(x, z)) p^1(\{\tau_1\}, dz).$$

By virtue of Lemma 4.9, there is a constant $N = N(d_1, d_2, \theta, \theta', \zeta_{\tau_1}(z), \beta')$

$$|u_{\tau_1-} \circ \tilde{H}_{\tau_1}^1(z) \cdot r_1^{-\xi_{\tau_1}(z)(\theta \vee \theta' + \epsilon + \beta')}|_{\beta'} \leq N |r_1^{-\theta \vee \theta' - \epsilon} u_{\tau_1-}^l|_{\beta'},$$

and hence

$$|r_1^{-\lambda_1} u_{\tau_1}(x)|_{\beta'} \leq N |r_1^{-\theta \vee \theta' - \epsilon} u_{\tau_1-}^l|_{\beta'} + \zeta_{\tau_1}(z),$$

where

$$\lambda_1 = (\xi_{\tau_1}(z)(\theta \vee \theta' + 1 + \epsilon + \beta')) \vee \theta \vee (\theta \vee \theta' + \epsilon).$$

We then proceed inductively, each time making use of the estimate (3.26), to obtain a unique solution $u = u(\tau)$ of (3.23), and hence (1.1), in $\mathfrak{C}^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$. This completes the proof of Theorem 2.5. \square

4 Appendix

4.1 Martingale and point measure measure moment estimates

Set $(Z, \mathcal{Z}, \pi) = (Z^1, \mathcal{Z}^1, \pi^1)$, $p(dt, dz) = p^1(dt, dz)$, and $q(dt, dz) = q^1(dt, dz)$. We will make use of the following moment estimates to derive the estimates of Γ_t and Ψ_t in Lemma 3.2. The notation $a \underset{p}{\sim} b$ is used to indicate that the quantity a is bounded above and below by a constant depending only on p times b .

Lemma 4.1. *Let $h : \Omega \times [0, T] \times Z \rightarrow \mathbf{R}^{d_1}$ be $\mathcal{P}_T \otimes \mathcal{Z}$ -measurable*

(1) *For each stopping time $\tau \leq T$ and $p \geq 2$,*

$$\begin{aligned} \mathbf{E} \left[\sup_{t \leq \tau} \left| \int_{[0, t]} \int_Z h_s(z) q(ds, dz) \right|^p \right] &\underset{p}{\sim} \mathbf{E} \left[\int_{[0, \tau]} \int_Z |h_s(z)|^p \pi(dz) ds \right] \\ &+ \mathbf{E} \left[\left(\int_{[0, \tau]} \int_Z |h_s(z)|^2 \pi(dz) ds \right)^{p/2} \right]. \end{aligned}$$

(2) *For each stopping time $\tau \leq T$ and $p \geq 1$,*

$$\begin{aligned} \mathbf{E} \left[\sup_{t \leq \tau} \left(\int_{[0, t]} \int_Z |h_s(z)| p(ds, dz) \right)^p \right] &\underset{p}{\sim} \mathbf{E} \left[\int_{[0, \tau]} \int_Z |h_s(z)|^p \pi(dz) ds \right] \\ &+ \mathbf{E} \left[\left(\int_{[0, \tau]} \int_Z |h_s(z)| \pi(dz) ds \right)^p \right], \end{aligned}$$

Proof. We will only prove part (2), since part (1) is well-known (see, e.g., [Kun04]) and it follows from (2) by the Burkholder-Davis-Gundy inequality. Assume that $h_t(\omega, z) > 0$ for all ω, t and z . Let

$$A_t = \int_{[0, t]} \int_Z h_s(z) p(ds, dz) \quad \text{and} \quad L_t = \int_{[0, t]} \int_Z h_s(z) \pi(dz) ds, \quad t \leq T.$$

It suffices to prove the claim for $p > 1$, since the case $p = 1$ is obvious. Fix an arbitrary stopping time $\tau \leq T$ and $p > 1$. For all ω and t , we have

$$A_t^p = \sum_{s \leq t} [(A_{s-} + \Delta A_s)^p - A_{s-}^p].$$

Thus, using the inequality

$$b^p \leq (a + b)^p - a^p \leq p(a + b)^{p-1}b \leq p2^{p-1}[a^{p-1}b + b^p], \quad a, b \geq 0,$$

for all ω and t , we get

$$A_t^p \leq p2^{p-2} \left[\int_0^t \int_Z A_{s-}^{p-1} h_s(z) p(ds, dz) + \int_{[0,t]} \int_h h_s(z)^p p(ds, dz) \right].$$

and

$$A_t^p \geq \int_{[0,t]} \int_Z h_s(z)^p p(ds, dz).$$

Then since A_t is an increasing process, we have

$$\mathbf{E} \int_{[0,\tau]} \int_Z h_s(z)^p p(ds, dz) \leq \mathbf{E} A_\tau^p \leq p2^{p-2} \mathbf{E} \left[A_\tau^{p-1} L_\tau + \int_{[0,\tau]} \int_Z h_s(z)^p p(ds, dz) \right].$$

It is easy to see that

$$\mathbf{E} L_\tau^p = p \mathbf{E} \int_{[0,\tau]} L_s^{p-1} dL_s = p \mathbf{E} \int_{[0,\tau]} L_s^{p-1} dA_s \leq p \mathbf{E} [L_\tau^{p-1} A_\tau].$$

Applying Young's inequality, for all $\varepsilon > 0$, \mathbf{P} -a.s.,

$$A_\tau^{p-1} L_\tau \leq \varepsilon A_\tau^p + \frac{(p-1)^{p-1}}{\varepsilon^{p-1} p^p} L_\tau^p \quad \text{and} \quad L_\tau^{p-1} A_\tau \leq \varepsilon L_\tau^p + \frac{(p-1)^{p-1}}{\varepsilon^{p-1} p^p} A_\tau^p.$$

Combining the above estimates, for any $\varepsilon_1 \in (0, \frac{1}{p})$, we have

$$\left(\frac{\varepsilon_1^{p-1} p^p (1 - p\varepsilon_1)}{p(p-1)^{p-1}} \mathbf{E} L_\tau^p \right) \vee \mathbf{E} \int_{[0,\tau]} \int_Z h_s(z)^p p(ds, dz) \leq \mathbf{E} A_\tau^p.$$

and for any $\varepsilon_2 \in (0, \frac{1}{p2^{p-2}})$

$$\mathbf{E} A_\tau^p \leq \frac{p2^{p-2}}{(1 - p2^{p-2}\varepsilon_2)} \mathbf{E} \left[\int_{[0,\tau]} \int_Z h_s(z)^p p(ds, dz) + \frac{(p-1)^{p-1}}{\varepsilon_2^{p-1} p^p} L_\tau^p \right],$$

which completes the proof. \square

4.2 Optional projection

The following lemma concerning the optional projection plays an integral role in Section 3.4 and the proof of Theorem 2.2.

Lemma 4.2. (cf. Theorem 1 in [Mey76]) *Let \mathcal{X} be a Polish space and $D([0, T]; \mathcal{X})$ be the space of \mathcal{X} -valued càdlàg trajectories with the Skorokhod \mathcal{J}_1 -topology. If \mathfrak{A} is a random variable taking values in $D([0, T]; \mathcal{X})$, then there exists a family of $\mathcal{B}([0, T]) \times \mathcal{F}$ -measurable non-negative measures $E^t(dU)$, $(\omega, t) \in \Omega \times [0, T]$, on $D([0, T]; \mathcal{X})$ and a random-variable ζ satisfying $\mathbf{P}(\zeta < T) = 0$ such that $E^t(D([0, T]; \mathcal{X})) = 1$ for $t < \zeta$ and $E^t(D([0, T]; \mathcal{X})) = 0$ for $t \geq \zeta$. In addition, E^t is càdlàg in the topology of weak convergence, $E^t = E^{t+}$ for all $t \in [0, T]$, and for each continuous and bounded functional F on $D([0, T]; \mathcal{X})$, the process $E^t(F)$ is the càdlàg version of $\mathbf{E}[F(\mathfrak{A})|\mathcal{F}_t]$. If $G : \Omega \times [0, T] \times [0, T] \times D([0, T]; \mathcal{X}) \rightarrow \mathbf{R}^{d_2}$ is bounded and $\mathcal{O} \times \mathcal{B}([0, T]) \times \mathcal{B}(D([0, T]; \mathcal{X}))$ -measurable, then*

$$\int_{D([0, T]; \mathcal{X})} G_t(\omega, t, U) E^t(dU) = E^t(G_t)$$

is the optional projection of $G_t(\mathfrak{A}) = G_t(\omega, t, \mathfrak{A})$. Furthermore, if $G = G_t(\omega, t, U)$ is bounded and $\mathcal{P} \times \mathcal{B}([0, T]) \times \mathcal{B}(D([0, T]; \mathcal{X}))$ -measurable, then $E^{t-}(G_t)$ is the predictable projection of $G_t(\mathfrak{A}) = G_t(\omega, t, \mathfrak{A})$.

Proof. We follow the proof of Theorem 1 in [Mey76]. Since $D([0, T]; \mathcal{X})$ is a Polish space, for each $t \in [0, T]$, there is family of probability measures $\tilde{E}_\omega^t(dw)$, $\omega \in \Omega$, on $D([0, T]; \mathcal{X})$ such that for each $A \in \mathcal{B}(D([0, T]; \mathcal{X}))$, $\tilde{E}^t(A)$ is \mathcal{F}_t -measurable and \mathbf{P} -a.s. ,

$$\mathbf{P}(\mathfrak{A} \in A | \mathcal{F}_t) = \tilde{E}^t(A).$$

For each $\omega \in \Omega$, let $I(\omega)$ be the set of all $t \in (0, T]$ such that for each bounded continuous function F on $D([0, T]; \mathcal{X})$, the function

$$r \mapsto \tilde{E}_\omega^r(F) = \int_{D([0, T]; \mathcal{X})} F(w) \tilde{E}^r(dw)$$

has a right-hand limit on $[0, s) \cap \mathbf{Q}$ and a left-hand limit on $(0, s] \cap \mathbf{Q}$ for every rational $s \in [0, T] \cap \mathbf{Q}$. Let $\zeta(\omega) = \sup(t : t \in I(\omega)) \wedge T$. It is easy to see that $\mathbf{P}(\zeta < T) = 0$. We set $\tilde{E}_\omega^t = 0$ if $\zeta(\omega) < t \leq T$. The function \tilde{E}_ω^t has left-hand and right-hand limits for all $t \in \mathbf{Q} \cap [0, T]$. We define $E_\omega^t = \tilde{E}_\omega^{t+}$ for each $t \in [0, T]$ (the limit is taken along the rationals), and E_ω^T is the left-hand limit at T along the rationals. The statement follows by repeating the proof of Theorem 1 in [Mey76] in an obvious way. \square

4.3 Estimates of Hölder continuous functions

In the coming lemmas, we establish some properties of weighted Hölder spaces that are used Section 3.5 and the proof of Theorem 2.5.

Lemma 4.3. *Let $\beta \in (0, 1]$ and $\theta_1, \theta_2 \in \mathbf{R}$ with $\theta_1 - \theta_2 \leq \beta$.*

(1) *There is a constant $c_1 = c_1(\theta_2, \beta)$ such that for all $\phi : \mathbf{R}^{d_1} \rightarrow \mathbf{R}$ with $|r_1^{-\theta_1} \phi|_0 + [r_1^{-\theta_2} \phi]_\beta =: N_1 < \infty$,*

$$|\phi(x) - \phi(y)| \leq c_1 N_1 (r_1(x)^{\theta_2} \vee r_1(y)^{\theta_2}) |x - y|^\beta,$$

for all $x, y \in \mathbf{R}^{d_1}$.

(2) *Conversely, if $\phi : \mathbf{R}^{d_1} \rightarrow \mathbf{R}$ satisfies $|r_1^{-\theta_1} \phi|_0 < \infty$ and there is a constant N_2 such that for all $x, y \in \mathbf{R}^{d_1}$,*

$$|\phi(x) - \phi(y)| \leq N_2 (r_1(x)^{\theta_2} \vee r_1(y)^{\theta_2}) |x - y|^\beta,$$

then

$$[r_1^{-\theta_2} \phi]_\beta \leq c_1 |r_1^{-\theta_1} \phi|_0 + N_2.$$

Proof. (1) For all x, y with $r_1(x)^{\theta_2} \geq r_1(y)^{\theta_2}$, we have

$$\begin{aligned} |\phi(x) - \phi(y)| &\leq r_1(x)^{\theta_2} [r_1^{-\theta_2} \phi]_\beta |x - y|^\beta + r_1(y)^{\theta_1 - \theta_2} |r_1^{-\theta_1} \phi|_0 |r_1^{\theta_2}(x) - r_1(y)^{\theta_2}| \\ &\leq ([r_1^{-\theta_2} \phi]_\beta + c_1 |r_1^{-\theta_1} \phi|_0) r_1(x)^{\theta_2} |x - y|^\beta, \end{aligned}$$

where $c_1 := 1 + \sup_{t \in (0, 1)} \frac{1-t^{\theta_2}}{(1-t)^\beta}$ if $\theta_2 \geq 0$ and $c_1 := 1 + \sup_{t \in (1, \infty)} \frac{(t^{\theta_2} - 1)^\beta}{(t-1)^\beta}$ if $\theta_2 < 0$, which proves the first claim. (2) For all x and y with $r_1(x)^{\theta_2} > r_1(y)^{\theta_2}$, we have

$$\begin{aligned} &|r_1(x)^{-\theta_2} \phi(x) - r_1(y)^{-\theta_2} \phi(y)| \\ &\leq r_1(x)^{-\theta_2} |\phi(x) - \phi(y)| + r_1(y)^{\theta_1 - \theta_2} |r_1^{-\theta_1}(y) \phi(y)| |r_1(y)^{\theta_2} r_1(x)^{-\theta_2} - 1| \\ &\leq (c_1 |r_1^{-\theta_1} \phi|_0 + N_2) |x - y|^\beta, \end{aligned}$$

which proves the second claim. \square

Lemma 4.4. *Let $\beta, \mu \in (0, 1]$ and $\theta_1, \theta_2, \theta_3, \theta_4 \in \mathbf{R}$ with $\theta_1 - \theta_2 \leq \beta$, $\theta_3 - \theta_4 \leq \mu$, and $\theta_3 \geq 0$. If $\phi : \mathbf{R}^{d_1} \rightarrow \mathbf{R}$ and $H : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_1}$ are such that*

$$|r_1^{-\theta_1} \phi|_0 + [r_1^{-\theta_2} \phi]_\beta =: N_1 < \infty \quad \text{and} \quad |r_1^{-\theta_3} H|_0 + [r_1^{-\theta_4} H]_\mu =: N_2 < \infty,$$

then

$$|\phi \circ H \cdot r_1^{-\theta_1 \theta_3}|_0 \leq |r_1^{-\theta_1} \phi|_0 (1 + |r_1^{-\theta_3} H|_0) \leq N_1 (1 + N_2)^{\theta_1}$$

and there is a constant $N = N(\beta, \mu, \theta_1, \theta_2)$ such that

$$[\phi \circ H \cdot r_1^{-\theta_2 \theta_3 - \beta \theta_4}]_{\beta \mu} \leq N N_1 (1 + N_2)^{\theta_2 + \beta}.$$

Proof. For each x , we have

$$r_1(H(x)) \leq (1 + |r_1^{-\theta_3} H|_0) r_1(x)^{\theta_3} \leq (1 + N_2) r_1(x)^{\theta_3},$$

and hence

$$|\phi \circ H \cdot r_1^{-\theta_1 \theta_3}|_0 \leq |r_1^{-\theta_1} \phi|_0 |r_1^{\theta_1} \circ H \cdot r_1^{-\theta_1 \theta_3}|_0 \leq N_1 (1 + N_2)^{\theta_1}.$$

Using Lemma 4.3, for all x and y , we get

$$\begin{aligned} |\phi(H(x)) - \phi(H(y))| &\leq NN_1(r_1(H(x)) \vee r_1(H(y)))^{\theta_2} |H(x) - H(y)|^\beta \\ &\leq NN_1(1 + N_2)^{\theta_2}(r_1(x) \vee r_1(y))^{\theta_2\theta_3} N_2^\beta (r_1(x) \vee r_1(y))^{\beta\theta_4} |x - y|^{\beta\mu} \\ &\leq NN_1(1 + N_2)^{\theta_2+\beta}(r_1(x) \vee r_1(y))^{\theta_2\theta_3+\beta\theta_4} |x - y|^{\beta\mu}, \end{aligned}$$

for some constant $N = N(\beta, \mu, \theta_1, \theta_2)$. Noting that

$$\theta_1\theta_3 - \theta_2\theta_3 - \beta\theta_4 = (\theta_1 - \theta_2)\theta_3 - \beta\theta_4 \leq \beta(\theta_3 - \theta_4) \leq \beta\mu,$$

we apply Lemma 4.3 to complete the proof. \square

Remark 4.5. Let $\beta \in (0, 1]$ and $\theta_1, \theta_2 \in \mathbf{R}$. Then there is a constant $N = N(\beta, \theta_1, \theta_2)$ such that for all $\phi : \mathbf{R}^{d_1} \rightarrow \mathbf{R}$ with $|r_1^{-\theta_1}\phi|_0 + [r_1^{-\theta_2}\phi]_\beta =: N_1 < \infty$, we have $|r^{-\theta}\phi|_\beta \leq NN_1$, where $\theta = \max\{\theta_1, \theta_2\}$. In particular, if in Lemma 4.4, $\theta_1 = \theta_2$ and $\theta_4 \geq 0$, then

$$|\phi \circ H \cdot r^{-\theta_1\theta_3-\beta\theta_4}|_{\beta\mu} \leq NN_1(1 + N_2)^{\theta_1+\beta}.$$

Proof. If $\theta_2 \geq \theta_1$, then the claim is obvious and if $\theta_1 > \theta_2$, for all x and y , we have

$$\begin{aligned} |r_1(x)^{-\theta_1}\phi(x) - r_1(y)^{-\theta_1}\phi(y)| &\leq r_1(x)^{\theta_2-\theta_1} |r_1(x)^{-\theta_2}\phi(x) - r_1(y)^{-\theta_2}\phi(y)| \\ &\quad + \left| \frac{r(y)^{\theta_1-\theta_2}}{r(x)^{\theta_1-\theta_2}} - 1 \right| |r_1^{-\theta_1}\phi|_0 \leq N_1(1 + c_1)|x - y|^\beta, \end{aligned}$$

where $c_1 := \sup_{t \in (0,1)} \frac{1-t^{\theta_1-\theta_2}}{(1-t)^\beta}$. \square

Lemma 4.6. For each $\theta \geq 0$ and $\beta > 1$, there are constants $N_1 = N_1(d_1, \theta, \beta)$ and $N_2(d_1, \theta, \beta)$ such that for all $\phi : \mathbf{R}^{d_1} \rightarrow \mathbf{R}$,

$$N_1|r_1^{-\theta}\phi|_\beta \leq \sum_{|\gamma| \leq [\beta]^-} |r_1^{-\theta}\partial^\gamma\phi|_0 + \sum_{|\gamma| = [\beta]^-} |r_1^{-\theta}\partial^\gamma\phi|_{[\beta]^+} \leq N_2|r_1^{-\theta}\phi|_\beta. \quad (4.1)$$

Proof. For each multi-index γ with $|\gamma| \leq [\beta]^-$ and x , we have

$$\partial^\gamma(r_1^{-\theta}\phi)(x) = \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ |\gamma_1| \geq 1}} r_1(x)^\theta \partial^{\gamma_1}(r_1^{-\theta})(x) r_1(x)^{-\theta} \partial^{\gamma_2}\phi(x) + r_1(x)^{-\theta} \partial^\gamma\phi(x).$$

It is easy to show by induction that for all multi-indices γ , $|r_1^\theta \partial^\gamma(r_1^{-\theta})|_1 < \infty$. Moreover, for all multi-indices γ with $|\gamma| < [\beta]^-$,

$$|r_1^{-\theta}\partial^\gamma\phi|_1 \leq |\nabla(r_1^{-\theta}\partial^\gamma\phi)| \leq |r_1^{-\theta}\nabla(r_1^{-\theta})|_0 |r_1^{-\theta}\partial^\gamma\phi|_0.$$

Thus, for each multi-index γ with $|\gamma| \leq [\beta]^-$,

$$|\partial^\gamma(r_1^{-\theta}\phi)|_0 \leq \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ |\gamma_1| \geq 1}} |r_1^\theta \partial^{\gamma_1}(r_1^{-\theta})|_0 |r_1^{-\theta}\partial^{\gamma_2}\phi|_0 + |r_1^{-\theta}\partial^\gamma\phi|_0$$

and for each multi-index γ with $|\gamma| = [\beta]^-$,

$$|\partial^\gamma(r_1^{-\theta}\phi)|_{[\beta]^+} \leq \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ |\gamma_1| \geq 1}} |r_1^\theta \partial^{\gamma_1}(r_1^{-\theta})|_1 |r_1^{-\theta} \nabla(r_1^{-\theta})|_0 |r_1^{-\theta} \partial^{\gamma_2} \nabla \phi|_0 + |r_1^{-\theta} \partial^\gamma \phi|_0.$$

This proves the leftmost inequality in (4.1). For all $i \in \{1, \dots, d\}$ and x ,

$$r_1^{-\theta} \partial_i \phi(x) = \partial_i(r_1^{-\theta} \phi)(x) - r_1(x)^{-\theta} \phi(x) r_1(x)^\theta \partial_i(r_1^{-\theta})(x).$$

It follows by induction that for all multi-indices γ with $|\gamma| \leq [\beta]^-$ and x , $r_1^{-\theta} \partial^\gamma \phi(x)$ is a sum of $\partial^\gamma(r_1^{-\theta} \phi)(x)$, a finite sum of terms, each of which is a product of one term of the form $\partial^{\tilde{\gamma}}(r_1^{-\theta} \phi)(x)$, $|\tilde{\gamma}| < |\gamma|$, and a finite number of terms of the form $\partial^{\gamma_1}(r_1^\theta) \partial^{\gamma_2}(r_1^{-\theta})$, $|\gamma_1|, |\gamma_2| \leq |\gamma|$. Since for all multi-indices γ_1 and γ_2 , we have $|\partial^{\gamma_1}(r_1^\theta) \partial^{\gamma_2}(r_1^{-\theta})|_1 < \infty$, the rightmost inequality in (4.1) follows. \square

Corollary 4.7. *For each $\theta \geq 0$ and $\beta > 1$, there are constants $N_1 = N_1(d_1, \theta, \beta)$ and $N_2(d_1, \theta, \beta)$ such that for all $\phi : \mathbf{R}^{d_1} \rightarrow \mathbf{R}$,*

$$N_1 |r_1^{-\theta} \phi|_\beta \leq |r_1^{-\theta} \phi|_0 + \sum_{|\gamma|=[\beta]^-} |r_1^{-\theta} \partial^\gamma \phi|_{[\beta]^+} \leq N_2 |r_1^{-\theta} \phi|_\beta.$$

Proof. It is well known that for an arbitrary unit ball $B \subset \mathbf{R}^{d_1}$ and any $1 \leq k < [\beta]^-$, there is a constant N such that for any $\varepsilon > 0$,

$$\sup_{x \in B, |\gamma|=k} |\partial^\gamma \phi| \leq N(\varepsilon \sup_{x \in B, |\gamma|=[\beta]^-} |\partial^\gamma \phi(x)| + \varepsilon^{-k} \sup_{x \in B} |\phi(x)|).$$

Let $U_0 = \{x \in \mathbf{R}^{d_1} : |x| \leq 1\}$ and $U_j = \{x \in \mathbf{R}^{d_1} : 2^{-j-1} \leq |x| \leq 2^j\}$, $j \geq 1$. For each j , we have

$$\begin{aligned} \sup_{x \in U_j, |\gamma|=k} |\partial^\gamma \phi(x)| &= \sup_{B \subseteq U_j} \sup_{x \in B, |\gamma|=k} |\partial^\gamma \phi(x)| \leq N(\varepsilon \sup_{B \subseteq U_j} \sup_{x \in B, |\gamma|=[\beta]^-} |\partial^\gamma \phi(x)| + \varepsilon^{-k} \sup_{B \subseteq U_j} \sup_{x \in B} |\phi(x)|) \\ &\leq N(\varepsilon \sup_{x \in U_j, |\gamma|=[\beta]^-} |\partial^\gamma \phi(x)| + \varepsilon^{-k} \sup_{x \in U_j} |\phi(x)|). \end{aligned}$$

Since for every j ,

$$2^{-\theta/2} 2^{-j\theta} \sup_{x \in U_j, |\gamma|=k} |\partial^\gamma \phi(x)| \leq \sup_{x \in U_j, |\gamma|=k} |r^{-\theta} \partial^\gamma \phi(x)| \leq 2^\theta 2^{-(j-1)\theta} \sup_{x \in U_j, |\gamma|=k} |\partial^\gamma \phi(x)|,$$

we see that

$$\begin{aligned} 2^{-\theta/2} \sup_j 2^{-j\theta} \sup_{x \in U_j, |\gamma|=k} |\partial^\gamma \phi(x)| &\leq \sup_j \sup_{x \in U_j, |\gamma|=k} |r^{-\theta} \partial^\gamma \phi(x)| = |r^{-\theta} \partial^\gamma \phi|_0 \\ &\leq 2^\theta \sup_j 2^{-j\theta} \sup_{x \in U_j, |\gamma|=k} |\partial^\gamma \phi(x)|, \end{aligned}$$

and the statement follows. \square

Remark 4.8. If $\phi : \mathbf{R}^{d_1} \rightarrow \mathbf{R}$ is such that $|r^{-\theta_1}\phi|_0 + |r^{-\theta_2}\nabla\phi|_0 < \infty$ for $\theta_1, \theta_2 \in \mathbf{R}$ with $\theta_1 - \theta_2 \leq 1$, then

$$[r^{-\theta_2}\phi]_1 \leq N(|r^{-\theta_1}\phi|_0 + |r^{-\theta_2}\nabla\phi|_0)$$

Proof. Indeed, for each x and y , we have

$$|\phi(x) - \phi(y)| \leq |r^{-\theta_2}\nabla\phi|_0 \int_0^1 r^{\theta_2}(x + s(y-x))ds|y-x| \leq |r^{-\theta_2}\nabla\phi|_0(r(y)^{\theta_2} \vee r(x)^{\theta_2})|y-x|,$$

and hence the claim follows from Lemma 4.3. \square

Lemma 4.9. *Let $n \in \mathbf{N}$, $\beta, \mu \in (0, 1]$, $\theta_3, \theta_4 \geq 0$ be such that $\theta_3 - \theta_4 \leq 1$. There is a constant $N = N(d_1, \theta_1, \theta_3, \theta_4, n, \beta)$ such that for all $\phi : \mathbf{R}^{d_1} \rightarrow \mathbf{R}$ with $r_1^{-\theta_1}\phi \in C^{n+\beta}(\mathbf{R}^{d_1}, \mathbf{R}^{d_1})$ and $H : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_1}$ with*

$$|r_1^{-\theta_3}H|_0 + |r_1^{-\theta_4}\nabla H|_{n-1+\mu} =: N_2 < \infty,$$

we have

$$|\phi \circ H \cdot r^{-\theta_1\theta_3}|_0 \leq |r_1^{-\theta_1}\phi|_0(1 + |r_1^{-\theta_3}H|_0)^{\theta_1}$$

and

$$|r_1^{-\theta_1\theta_3-\theta_4(n+\mu\wedge\beta)}\nabla(\phi \circ H)|_{n-1+\mu\wedge\beta} \leq N|r_1^{-\theta_1}\phi|_{n+\beta}(1 + N_2)^{\theta_1+\mu\wedge\beta+n}.$$

Proof. It follows immediately from Lemma 4.4 and Remark 4.8 that

$$|\phi \circ H \cdot r^{-\theta_1\theta_3}|_0 \leq |r_1^{-\theta_1}\phi|_0(1 + |r_1^{-\theta_3}H|_0)^{\theta_1}.$$

Using induction, we get that for each x and $|\gamma| = n$,

$$\partial^\gamma(\phi(H(x))) = \mathcal{I}_1^\gamma(x) + \mathcal{I}_2^\gamma(x) + \mathcal{I}_3^\gamma(x),$$

where

$$\mathcal{I}_1^\gamma(x) = \sum_{i=1}^{d_1} \partial_i \phi(H(x)) \partial^\gamma H^i(x)$$

$\mathcal{I}_2^\gamma(x)$ is a finite sum of terms of the form

$$\partial_{i_1} \cdots \partial_{i_{|\gamma|}} \phi(H(x)) \partial^{\tilde{\gamma}_1} H^{i_1} \cdots \partial^{\tilde{\gamma}_{|\gamma|}} H^{i_{|\gamma|}}$$

with $i_1, \dots, i_{|\gamma|} \in \{1, 2, \dots, d\}$, $|\tilde{\gamma}_1| = \cdots = |\tilde{\gamma}_{|\gamma|}| = 1$, and $\sum_{k=1}^{|\gamma|} \tilde{\gamma}_k = \gamma$, if $n \geq 2$ and zero otherwise, and where $\mathcal{I}_3^\gamma(x)$ is a finite sum of terms of the form

$$\partial_{i_1} \cdots \partial_{i_k} \phi(H(x)) \partial^{\tilde{\gamma}_1} H^{i_1}(x) \cdots \partial^{\tilde{\gamma}_k} H^{i_k}(x)$$

with $2 \leq k < n$, $i_1, i_2, \dots, i_k \in \{1, \dots, d\}$, and $\sum_{j=1}^k \tilde{\gamma}_j = \gamma$, $1 \leq |\tilde{\gamma}_j| < |\gamma|$, if $n \geq 3$, and zero otherwise. Thus, owing to Lemmas 4.4 and 4.6, for any multi-index γ with $|\gamma| = n$, we have

$$|r_1^{-\theta_3\theta_1-\theta_4}\mathcal{I}_1^\gamma|_0 \leq N|r_1^{-\theta_1}\nabla\phi|_0(1 + |r_1^{-\theta_3}H|_0)^{\theta_1}|r_1^{-\theta_4}\partial^\gamma H|_0,$$

$$|r_1^{-\theta_3\theta_1-n\theta_4}\mathcal{I}_2^\gamma|_0 \leq N|r_1^{-\theta_1}\partial^\gamma\phi|_0(1+|r_1^{-\theta_3}H|_0)^{\theta_1}|r_1^{-\theta_4}\nabla H|_0^n,$$

and

$$|r_1^{-\theta_3\theta_1-(n-1)\theta_4}\mathcal{I}_3^\gamma|_0 \leq N|r_1^{-\theta_1}\phi|_{n-1}(1+|r_1^{-\theta_3}H|_0+|r_1^{-\theta_4}\nabla H|)^{\theta_1+n-1},$$

and hence

$$|r_1^{-\theta_1\theta_3-n\theta_4}\partial^\gamma(\phi \circ H)|_0 \leq N|r_1^{-\theta_1}\phi|_n(1+|r_1^{-\theta_3}H|_0+|r_1^{-\theta_4}\nabla H|)^{\theta_1+n}.$$

Once again appealing to Lemmas 4.4 and 4.6, for all multi-indices γ with $|\gamma| = n$, we get

$$\begin{aligned} |r_1^{-\theta_1\theta_3-(1+\mu\wedge\beta)\theta_4}\mathcal{I}_1^\gamma|_{\mu\wedge\beta} &\leq N|r_1^{-\theta_1}\phi|_{1+\mu\wedge\beta}(1+N_2)^{\theta_1+\mu\wedge\beta+1}, \\ |r_1^{-\theta_1\theta_3-(n+\mu\wedge\beta)\theta_4}\mathcal{I}_2^\gamma|_{\mu\wedge\beta} + |r_1^{-\theta_1\theta_3-(n-1+\mu\wedge\beta)\theta_4}\mathcal{I}_3^\gamma|_{\mu\wedge\beta} &\leq N|r_1^{-\theta_1}\phi|_{n+\mu\wedge\beta}(1+N_2)^{\theta_1+n+\mu\wedge\beta}. \end{aligned}$$

Then applying Lemmas 4.4 and 4.6, we complete the proof. \square

We shall now provide some useful estimates of composite functions of diffeomorphisms.

Lemma 4.10. *Let $H : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_1}$ be continuously differentiable and assume that for all $x \in \mathbf{R}^{d_1}$,*

$$|H(x)| \leq L_0 + L_1|x| \quad \text{and} \quad |\nabla H(x)| \leq L_2.$$

Assume that for all $x \in \mathbf{R}^{d_1}$, $\kappa(x) = (I_{d_1} + \nabla H(x))^{-1}$ exists and $|\kappa(x)| \leq N_\kappa$.

(1) *Then the mapping $\tilde{H}(x) := x + H(x)$ is a diffeomorphism with $\tilde{H}^{-1}(x) = x - H(\tilde{H}^{-1}(x)) =: x + F(x)$ and for all $x \in \mathbf{R}^{d_1}$,*

$$|F(x)| \leq L_0 + L_1L_0N_\kappa + L_1N_\kappa|x|, \quad |\nabla F(x)| \leq N_\kappa L_2, \quad |(I_{d_1} + \nabla F(x))^{-1}| \leq 1 + L_2.$$

For all $p \in \mathbf{R}$, there is a constant $N = N(L_0, L_1, N_\kappa, p)$ such that for all $x \in \mathbf{R}^{d_1}$,

$$\frac{r_1^p(\tilde{H}(x))}{r_1^p(x)} + \frac{r_1^p(\tilde{H}^{-1}(x))}{r_1^p(x)} \leq N, \quad r_1^{-1}(x)|H^i(x) + F^{k;i}(x)| \leq N[H]_1|r_1^{-1}H|_0.$$

Moreover, there is a constant $N = N(L_0, L_1, N_\kappa, p)$ such that

$$\begin{aligned} \left| \frac{r_1^p(\tilde{H})}{r_1^p} - 1 + \mathbf{1}_{(1,2]}(\alpha)pH^i r_1^{-2}x^j \right|_\alpha + \left| \frac{r_1^p(\tilde{H}^{-1})}{r_1^p} - 1 - \mathbf{1}_{(1,2]}(\alpha)pF^i r_1^{-2}x^j \right|_\alpha \\ \leq N(|r_1^{-1}H|_0^{[\alpha]^-+1} + [H]_1^{[\alpha]^-+1}). \end{aligned}$$

(2) *If for some $\beta > 1$, $|\nabla H|_{\beta-1} \leq L_3$, then there is a constant $N = N(d_1, \beta, N_\kappa, L_3)$ such that*

$$|\nabla F|_{\beta-1} \leq N|\nabla H|_{\beta-1}. \quad (4.2)$$

(3) *If for some $\beta \geq 1$, $|\nabla H|_{\beta-1} \leq L_3$, then for each $\theta \geq 0$, there is a constant $N = N(d_1, \beta, N_\kappa, L_1, L_3, \theta)$ such that*

$$\left| \frac{r_1^\theta \circ \tilde{H}^{-1}}{r_1^\theta} - 1 \right|_\beta \leq N[|r_1^{-1}H|_0 + |\nabla H|_{\beta-1}].$$

(4) If $|H|_0 \leq L_4$, and for some $\beta > 0$, $|\nabla H|_{\beta \vee 1-1} \leq L_5$ and $\phi : \mathbf{R}^{d_1} \rightarrow \mathbf{R}$ is such that for some $\mu \in (0, 1]$ and $\theta \geq 0$, $r_1^{-\theta} \phi \in C^{\beta+\mu}(\mathbf{R}^{d_1}; \mathbf{R})$, then there is a constant $N = N(d_1, \beta, \mu, N_\kappa, L_4, L_5, \theta)$ such that

$$\begin{aligned} |r_1^{-\theta}(\phi \circ \tilde{H}^{-1} - \phi)|_\beta &\leq N|r_1^{-\theta}\phi|_\beta(|H|_0 + |\nabla H|_{\beta \vee 1-1}) \\ &\quad + N\mathbf{1}_{(0,1]}(\{\beta\}^+ + \mu) \sum_{|\gamma|=[\beta]^-} [\partial^\gamma(r_1^{-\theta}\phi)]_{\{\beta\}^+ + \mu} L_4^\mu \\ &\quad + N\mathbf{1}_{(1,2]}(\{\beta\}^+ + \mu) \sum_{|\gamma|=[\beta]^-} \left([\nabla \partial^\gamma(r_1^{-\theta}\phi)]_{\{\beta\}^+ + \mu-1} L_4^\mu + |\nabla \partial^\gamma(r_1^{-\theta}\phi)|_0 |\nabla H|_0 \right). \end{aligned}$$

Proof. (1) Since $(I_{d_1} + \nabla H(x))^{-1}$ exists for each x , it follows from Theorem 0.2 in [DHI13] that the mapping \tilde{H} is a global diffeomorphism. For each x , we easily verify $\tilde{H}^{-1}(x) = x - H(\tilde{H}^{-1}(x))$ by substituting $\tilde{H}(x)$ into the expression. Simple computations show that for all x , we have

$$|\nabla \tilde{H}(x)| \leq 1 + L_2, \quad |\nabla \tilde{H}^{-1}(x)| = |\kappa(\tilde{H}^{-1}(x))| \leq N_\kappa, \quad |\nabla F(x)| = |\nabla H(\tilde{H}^{-1}(x)) \nabla \tilde{H}^{-1}(x)| \leq N_\kappa L_2,$$

and

$$|(I_{d_1} + \nabla F(x))^{-1}| = |\nabla \tilde{H}^{-1}(x)|^{-1} = |\kappa(\tilde{H}^{-1}(x))|^{-1} = |I_{d_1} + \nabla H(\tilde{H}^{-1}(x))| \leq 1 + L_2.$$

For all x and y , we easily obtain

$$|\tilde{H}(x) - \tilde{H}(y)| \leq (1 + L_2)|x - y|, \quad |\tilde{H}^{-1}(x) - \tilde{H}^{-1}(y)| \leq N_\kappa|x - y|,$$

and hence

$$N_\kappa^{-1}|x - y| \leq |\tilde{H}(x) - \tilde{H}(y)|, \quad (1 + L_2)^{-1}|x - y| \leq |\tilde{H}^{-1}(x) - \tilde{H}^{-1}(y)|. \quad (4.3)$$

Making use of (4.3), for all x , we get

$$N_\kappa^{-1}|x| \leq L_0 + |\tilde{H}(x)|, \quad |\tilde{H}^{-1}(x)| \leq N_\kappa L_0 + N_\kappa|x|, \quad |x| \leq L_0 + L_1|\tilde{H}^{-1}(x)|,$$

and thus

$$|F(x)| \leq L_0 + L_1 N_\kappa L_0 + L_1 N_\kappa |x|.$$

The rest of the estimates then follow easily from the above estimates and Taylor's theorem.

(2) Using the chain rule, for all x , we obtain

$$\nabla F(x) = -\nabla H(\tilde{H}^{-1}(x)) \nabla \tilde{H}^{-1}(x) = -\nabla H(\tilde{H}^{-1}(x)) \kappa(\tilde{H}^{-1}(x)), \quad (4.4)$$

and hence $|\nabla F|_0 \leq N_\kappa |\nabla H|_0$. For all x and y , we have

$$\kappa(\tilde{H}^{-1}(y)) - \kappa(\tilde{H}^{-1}(x)) = \kappa(y)[\nabla H(\tilde{H}^{-1}(x)) - \nabla H(\tilde{H}^{-1}(y))]\kappa(x),$$

and thus since $[\tilde{H}^{-1}]_1 \leq (1 + N_\kappa L_3)$ by part (1), we have for all $\delta \in (0, 1 \wedge \beta]$,

$$[\kappa(\tilde{H}^{-1})]_\delta \leq N_\kappa^2 (1 + N_\kappa L_3)^\delta [\nabla H]_\delta.$$

It follows that there is a constant $N = N(N_\kappa, L_3)$ such that for all $\delta \in (0, 1 \wedge \beta]$,

$$|\nabla F|_\delta \leq N|H|_\delta.$$

It is well-known that the inverse map \mathfrak{I} on the set of invertible $d_1 \times d_1$ matrices is infinitely differentiable and for each n , there exists a constant $N = N(n, d_1)$ such that for all invertible matrices A , the n -th derivative of \mathfrak{I} evaluated at A , denoted $\mathfrak{I}^{(n)}(A)$, satisfies

$$|\mathfrak{I}^{(n)}(A)| \leq N|A|^{-n-1} \leq N|A|^{-1|n+1|}.$$

Using induction, we find that for all multi-indices γ with $|\gamma| \leq [\beta]^-$ and for each x , $\partial^\gamma F(x)$ is a finite sum of terms, each of which is a finite product of

$$\partial^{\tilde{\gamma}} H(\tilde{H}^{-1}(x)), \quad \kappa(\tilde{H}^{-1}(x))^{\tilde{n}}, \quad \text{and} \quad \mathfrak{I}^{(\tilde{n}-1)}(I + \nabla H(\tilde{H}^{-1}(x))), \quad |\tilde{\gamma}| \leq |\gamma|, \quad \tilde{n} \in \{1, \dots, |\gamma|\}.$$

Therefore, differentiating (4.4) and estimating directly, we easily obtain (4.2).

(3) For each x , we have

$$\begin{aligned} \frac{r_1(\tilde{H}^{-1}(x))^\theta}{r_1(x)^\theta} - 1 &= r_1(x)^{-\theta} \int_0^1 r_1(G_s(x))^{\theta-2} G_s(x)^* F(x) ds \\ &= \int_0^1 \frac{r_1^{\theta-1}(G_s(x))}{r_1(x)^{\theta-1}} K(G_s(x))^* ds r_1(x)^{-1} F(x), \end{aligned}$$

where $G_s(x) := x + sF(x)$, $s \in [0, 1]$, and $J(x) := r_1(x)^{-1}x$. According to part (1) and (2), we have $|r_1^{-1}F|_0 \leq N|r_1^{-1}H|_0$ and $|\nabla F|_{\beta-1} \leq N|\nabla H|_{\beta-1}$, and hence

$$|r_1^{-1}G_s|_0 \leq N(1 + |r_1^{-1}H|_0), \quad |\nabla G_s(x)|_{\beta-1} \leq N(1 + |\nabla H|_{\beta-1}).$$

and

$$|J \circ G_s|_\beta \leq N(1 + |r_1^{-1}H|_0 + |\nabla H|_{\beta-1}),$$

for some constant N independent of s . Moreover, using Lemma 4.9, we find

$$|r_1^{\theta-1} \circ G_s \cdot r_1^{1-\theta}|_\beta \leq N \left(1 + |r_1^{-1}H|_0 + |\nabla H|_{\beta-1}\right)^{\theta+\beta}.$$

The statement then follows.

(4) First, we will consider the case $\theta = 0$. By part (1), we have for each $\bar{\mu} \in (0, (\beta + \mu) \wedge 1]$,

$$|\phi \circ \tilde{H}^{-1} - \phi|_0 \leq [\phi]_{\bar{\mu}} |H \circ \tilde{H}^{-1}|_0^\mu \leq [\phi]_{\bar{\mu}} |H|_0^{\bar{\mu}}.$$

First, let us consider the case $\beta \leq 1$. For each x , let $\mathcal{J}(x) = \phi(\tilde{H}^{-1}(x)) - \phi(x)$. For all x and y , it is clear that

$$|\mathcal{J}(x) - \mathcal{J}(y)| \leq A(x, y) + B(x, y) + C(x, y),$$

where

$$A(x, y) := |\mathcal{J}(x)|\mathbf{1}_{[L_4, \infty)}(|x - y|), \quad B(x, y) := |\mathcal{J}(y)|\mathbf{1}_{[L_4, \infty)}(|x - y|),$$

and

$$C(x, y) := |\mathcal{J}(x) - \mathcal{J}(y)|\mathbf{1}_{[0, L_4)}(|x - y|).$$

Moreover, owing to part (1), if $\beta + \mu \leq 1$, then for all x , and y , we have

$$\begin{aligned} A(x, y) &\leq [\phi]_{\beta+\mu} L_4^{\beta+\mu} \mathbf{1}_{[L_4, \infty)}(|x - y|) \leq [\phi]_{\beta+\mu} L_4^\mu |x - y|^{[\beta]^+}, \\ B(x, y) &\leq [\phi]_{\beta+\mu} L_4^\mu |x - y|^\beta, \end{aligned}$$

and

$$\begin{aligned} C(x, y) &\leq [\phi]_{\beta+\mu} [\tilde{H}^{-1}]_1^{\beta+\mu} |x - y|^{\beta+\mu} \mathbf{1}_{[0, L_4)}(|x - y|) + [\phi]_{\beta+\mu} |x - y|^{\beta+\mu} \mathbf{1}_{[0, L_4)}(|x - y|) \\ &\leq N[\phi]_{\beta+\mu} L_4^\mu |x - y|^\beta \end{aligned}$$

for some constant $N = N(\mu, N_\kappa, L_4)$. Using the identity

$$\begin{aligned} &\mathcal{J}(x) - \mathcal{J}(y) \\ &= - \int_0^1 \left(\nabla \phi \left(x - \theta H(\tilde{H}^{-1}(x)) \right) - \nabla \phi \left(y - \theta H(\tilde{H}^{-1}(y)) \right) \right) H(\tilde{H}^{-1}(x)) d\theta \\ &\quad - \int_0^1 \nabla \phi \left(y - \theta H(\tilde{H}^{-1}(y)) \right) (H(\tilde{H}^{-1}(y)) - H(\tilde{H}^{-1}(x))), \end{aligned}$$

and part (1), if $\beta + \mu > 1$, we get that there is a constant $N = N(\mu, N_\kappa, L_4)$ such that for all x and y ,

$$\begin{aligned} |\mathcal{J}(x) - \mathcal{J}(y)|\mathbf{1}_{[L_4, \infty)}(|x - y|) &\leq N([\nabla \phi]_{\beta+\mu-1} |x - y|^{\beta+\mu-1} L_4 + |\nabla \phi|_0 |x - y| [H]_1) \mathbf{1}_{[L_4, \infty)}(|x - y|) \\ &\leq N[\nabla \phi]_{\beta+\mu-1} L_4^\mu |x - y|^\beta + N|\nabla \phi|_0 |\nabla H|_0 |x - y|. \end{aligned}$$

Moreover, since

$$\begin{aligned} &\mathcal{J}(x) - \mathcal{J}(y) \\ &= \int_0^1 \nabla \phi \left(\tilde{H}^{-1}(x + \theta(y - x)) \right) \left(\nabla \tilde{H}^{-1}(x + \theta(y - x)) - I_d \right) (x - y) d\theta \\ &\quad + \int_0^1 \left(\nabla \phi \left(\tilde{H}^{-1}(x + \theta(y - x)) \right) - \nabla \phi \left(x + \theta(y - x) \right) \right) (x - y) d\theta, \end{aligned}$$

by part (1) and (4.2), if $\beta + \mu > 1$, we have that there is a constant $N = N(\mu, N_\kappa, L_4)$ such that for all x and y ,

$$\begin{aligned} |\mathcal{J}(x) - \mathcal{J}(y)|\mathbf{1}_{[0, L_4)}(|x - y|) &\leq (|\nabla \phi|_0 |\nabla H|_0 + [\nabla \phi]_{\beta+\mu-1} L_4^{\beta+\mu-1}) |x - y| \mathbf{1}_{[0, L_4)}(|x - y|) \\ &\leq |\nabla \phi|_0 |\nabla H|_0 |x - y| + [\nabla \phi]_{\beta+\mu-1} L_4^\mu |x - y|^\beta. \end{aligned}$$

Combining the above estimates, we get that for all $\beta \leq 1$ and $\mu \in (0, 1]$, there is a constant $N = N(\mu, N_\kappa, L_4)$ such that

$$[\phi \circ \tilde{H}^{-1} - \phi]_\beta \leq N \mathbf{1}_{[0,1]}(\beta + \mu) [\phi]_{\beta+\mu} L_4^\mu + N \mathbf{1}_{(1,2]}(\beta + \mu) ([\nabla \phi]_{\beta+\mu-1} + |\nabla \phi|_0 |\nabla H|_0). \quad (4.5)$$

This proves the desired estimate for $\beta \leq 1$ and $\theta = 0$. We now consider the case $\beta > 1$. For $\beta > 1$, it is straightforward to prove by induction that for all multi-indices γ with $1 \leq |\gamma| \leq [\beta]^-$ and for all x ,

$$\partial^\gamma(\phi(\tilde{H}^{-1}))(x) = \mathcal{J}_1^\gamma(x) + \mathcal{J}_2^\gamma(x) + \mathcal{J}_3^\gamma(x) + \mathcal{J}_4^\gamma(x),$$

where

$$\mathcal{J}_1^\gamma(x) := \partial^\gamma \phi(\tilde{H}^{-1}(x)),$$

$$\mathcal{J}_2^\gamma(x) = \partial^\gamma \phi(\tilde{H}^{-1})(\partial_1 \tilde{H}^{-1;1})^{\gamma_1} \dots (\partial_d \tilde{H}^{-1;d})^{\gamma_d} - 1,$$

$\mathcal{J}_3^\gamma(x)$ is a finite sum of terms of the form

$$\partial_{j_1} \dots \partial_{j_k} \phi(\tilde{H}^{-1}(x)) \partial^{\tilde{\gamma}_1} \tilde{H}^{-1;j_1}(x) \dots \partial^{\tilde{\gamma}_k} \tilde{H}^{-1;j_k}(x)$$

with $1 \leq k < [\beta]^-$, $j_1, \dots, j_k \in \{1, \dots, d\}$, and $\sum_{j=1}^k \tilde{\gamma}_j = \gamma$, and $\mathcal{J}_4(x)$ is a finite sum of terms of the form

$$\partial_{j_1} \dots \partial_{j_{[\beta]^-}} \phi(\tilde{H}^{-1}(x)) \partial_{i_1} \tilde{H}^{-1;j_1}(x) \dots \partial_{i_{[\beta]^-}} \tilde{H}^{-1;j_{[\beta]^-}}(x)$$

with $i_1, j_1, \dots, i_{[\beta]^-}, j_{[\beta]^-} \in \{1, \dots, d\}$ and at least one pair $i_k \neq j_k$. Since for each x ,

$$\nabla \tilde{H}^{-1}(x) = I + \nabla F(x)$$

and (4.2) holds, there is a constant $N = N(d_1, \beta)$ such that

$$\sum_{1 \leq |\gamma| \leq \beta} \sum_{i=2}^4 |\mathcal{J}_i^\gamma|_0 + \sum_{|\gamma|=\beta} \sum_{i=2}^4 |\mathcal{J}_i^\gamma|_{[\beta]^+} \leq N |\nabla \phi|_{\beta-1} |\nabla F|_{\beta-1} \leq N |\nabla \phi|_{\beta-1} |\nabla H|_{\beta-1}.$$

If $\beta > 2$, then for all multi-indices γ with $1 \leq |\gamma| < [\beta]^-$, we get

$$[\mathcal{J}_1^\gamma - \partial^\gamma \phi]_0 = |\partial^\gamma \phi \circ \tilde{H}^{-1} - \partial^\gamma \phi|_0 \leq [\partial^\gamma \phi]_1 |H|_0.$$

It is easy to see that there is a constant $N = N(L_4, N_\kappa)$ such that for all γ with $|\gamma| = [\beta]^-$ and all $\bar{\mu} \in (0, (\{\beta\}^+ + \mu) \wedge 1]$,

$$[\mathcal{J}_1^\gamma - \partial^\gamma \phi]_0 = |\partial^\gamma \phi \circ \tilde{H}^{-1} - \partial^\gamma \phi|_0 \leq [\partial^\gamma \phi]_{\bar{\mu}} |H|_0^{\bar{\mu}}.$$

Moreover, appealing to the estimate (4.5), we obtain

$$\begin{aligned} & [\mathcal{J}_1^\gamma - \partial^\gamma \phi]_{[\beta]^+} \\ & \leq N \mathbf{1}_{[0,1]}(\{\beta\}^+ + \mu) [\partial^\gamma \phi]_{[\beta]^+ + \mu} L_4^\mu + N \mathbf{1}_{(1,2]}(\{\beta\}^+ + \mu) ([\nabla \partial^\gamma \phi]_{[\beta]^+ + \mu - 1} + |\nabla \partial^\gamma \phi|_0 |\nabla H|_0). \end{aligned}$$

Let us now consider the case $\theta > 0$. The following decomposition obviously holds for all x :

$$r_1(x)^{-\theta} \phi(\tilde{H}^{-1}(x)) - r_1(x)^{-\theta} \phi(x) = \hat{\phi}(\tilde{H}^{-1}) - \hat{\phi}(x) + \left(\frac{r_1(\tilde{H}^{-1}(x))^\theta}{r_1(x)^\theta} - 1 \right) \hat{\phi}(\tilde{H}^{-1}(x)),$$

where $\hat{\phi} = r_1^{-\theta} \phi \in C^\beta(\mathbf{R}^{d_1}; \mathbf{R}^{d_1})$. Thus, to complete the proof we require

$$|\hat{\phi} \circ \tilde{H}^{-1}|_\beta \leq N |\hat{\phi}|_\beta \quad \text{and} \quad \left| \frac{r_1^\theta \circ \tilde{H}^{-1}}{r_1^\theta} - 1 \right|_\beta \leq N(|H|_0 + |\nabla H|_{\beta \vee 1-1}).$$

The latter inequality was proved in part (3) and the first inequality follows from part (2) and Lemma 4.9. \square

Remark 4.11. Let $H : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_1}$ be continuously differentiable and assume that for all x ,

$$|\nabla H(x)| \leq \eta < 1.$$

Then for each $x \in \mathbf{R}^{d_1}$,

$$|(I_{d_1} + \nabla H(x))^{-1}| \leq |I_{d_1} + \sum_{k=1}^{\infty} (-1)^k \nabla H(x)^k| \leq \frac{1}{1 - \eta}.$$

4.4 Stochastic Fubini theorem

Let $m = (m^\varrho)_{t \leq T}$, $\varrho \geq 1$, be a sequence of \mathbf{F} -adapted locally square integrable continuous martingales issuing from zero such that \mathbf{P} -a.s. for all $t \in [0, T]$, $\langle m^{\varrho_1}, m^{\varrho_2} \rangle_t = 0$ for $\varrho_1 \neq \varrho_2$ and $\langle m^\varrho \rangle_t = N_t$ for $\varrho \geq 1$, where N_t is a \mathcal{P}_T -measurable continuous increasing processes issuing from zero. Let $\eta(dt, dz)$ be a \mathbf{F} -adapted integer-valued random measure on $([0, T] \times E, \mathcal{B}([0, T]) \otimes \mathcal{E})$, where (U, \mathcal{U}) is a Blackwell space. We assume that $\eta(dt, dz)$ is optional, $\tilde{\mathcal{P}}_T$ -sigma-finite, and quasi-left continuous. Thus, there exists a unique (up to a \mathbf{P} -null set) dual predictable projection (or compensator) $\eta^p(dt, dz)$ of $\eta(dt, dz)$ such that $\mu(\omega, \{t\} \times U) = 0$ for all ω and t . We refer the reader to Ch. II, Sec. 1, in [JS03] for any unexplained concepts relating to random measures.

Let (X, Σ, μ) be a sigma-finite measure space; that is, there is an increasing sequence of Σ -measurable sets X_n , $n \in \mathbf{N}$, such that $X = \cup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$ for each n . Let $f : \Omega \times [0, T] \times X \rightarrow \mathbf{R}^{d_2}$ be $\mathcal{R}_T \otimes \Sigma$ -measurable, $g : \Omega \times [0, T] \times X \rightarrow \ell_2(\mathbf{R}^{d_2})$ be $\mathcal{R}_T \otimes \Sigma / \mathcal{B}(\ell_2(\mathbf{R}^{d_2}))$ -measurable, and $h : \Omega \times [0, T] \times X \times U \rightarrow \mathbf{R}^{d_2}$ be $\mathcal{P}_T \otimes \Sigma \otimes \mathcal{U}$ -measurable. Moreover, assume that for each $t \in [0, T]$ and $x \in X$, \mathbf{P} -a.s.

$$\int_{[0, T]} |g_t(x)|^2 dN_t + \int_{[0, T]} \int_U |h_t(x, z)|^2 \eta^p(dt, dz) < \infty.$$

Let $F = F_t(x) : \Omega \times [0, T] \times X \rightarrow \mathbf{R}^{d_2}$ be $\mathcal{O}_T \otimes \mathcal{B}(X)$ -measurable and assume that for $d\mathbf{P} \otimes \mu$ -almost all $(t, x) \in [0, T] \times X$,

$$F_t(x) = \int_{[0, t]} g_s^\varrho(x) dm_s^\varrho + \int_{[0, t]} \int_U h_s(x, z) \tilde{\eta}(dt, dz)$$

where $\tilde{\eta}(dt, dz) = \eta(dt, dz) - \eta^p(dt, dz)$.

The following version of the stochastic Fubini theorem is a straightforward extension of Lemma 2.6 [Kry11] and Corollary 1 in [Mik83]. See also Proposition 3.1 in [Zho13], Theorem 2.2 in [Ver12], and Theorem 1.4.8 in [Roz90]. Indeed, to prove it for a bounded measure, we can use a monotone class argument as in Theorem 64 in [Pro05]. To handle the general setting with possibly infinite μ , we use assumptions (ii) and (iii) below and take limits on the sets X_n using the Lenglart domination lemma (Theorem 1.4.5 on page 66 in [LS89]) and the following well known inequalities:

$$\mathbf{E} \sup_{t \leq T} \left| \int_{[0,t]} g_s^e dw_s^e \right| \leq N \mathbf{E} \left(\int_{[0,T]} |g_t(x)|^2 dw_t^e \right)^{1/2}$$

$$\mathbf{E} \sup_{t \leq T} \left| \int_{[0,t]} \int_U h_t(x, z) \tilde{\eta}(dt, dz) \right| \leq N \mathbf{E} \left(\int_{[0,T]} \int_U |h_t(x, z)|^2 \eta^p(dt, dz) \right)^{1/2},$$

where $\tau \leq T$ is an arbitrary stopping time and $N = N(T)$ is a constant independent of g and h .

Proposition 4.12 (c.f. Corollary 1 in [Mik83] and Lemma 2.6 in [Kry11]). *Assume that*

(1) **P**-a.s. for each $n \geq 1$,

$$\int_{X_n} \left(\int_{[0,T]} |g_t(x)|^2 dN_t \right)^{1/2} \mu(dx) + \int_{X_n} \left(\int_{[0,T]} \int_{U_1} |h_t(x, z)|^2 \eta^p(dt, dz) \right)^{1/2} \mu(dx) < \infty;$$

(2) **P**-a.s.

$$\int_{[0,T]} \left(\int_X |g_t(x)| \mu(dx) \right)^2 dt + \int_{[0,T]} \int_U \left(\int_X |h_t(x, z)| \mu(dx) \right)^2 \eta^p(dt, dz);$$

(3) **P**-a.s. for all $t \in [0, T]$,

$$\int_X |F_t(x)| \mu(dx) < \infty.$$

Then **P**-a.s. for all $t \in [0, T]$,

$$\int_X F_t(x) \mu(dx) = \int_{[0,t]} \int_X g_s^e(x) \mu(dx) dm_s^e + \int_{[0,t]} \int_U \int_X h_s(x, z) \mu(dx) \tilde{\eta}(dr, dz)$$

We obtain the following corollary by applying Minkowski's integral inequality.

Corollary 4.13. *Assume that **P**-a.s.*

$$\int_X \left(\int_{[0,T]} |g_t(x)|^2 dN_t \right)^{1/2} \mu(dx) + \int_X \left(\int_{[0,T]} \int_{U_1} |h_t(x, z)|^2 \eta^p(dt, dz) \right)^{1/2} \mu(dx) < \infty. \quad (4.6)$$

Then **P**-a.s. for all $t \in [0, T]$,

$$\int_X F_t(x) \mu(dx) = \int_{[0,t]} \int_X g_s^e(x) \mu(dx) dm_s^e + \int_{[0,t]} \int_U \int_X h_s(x, z) \mu(dx) \tilde{\eta}(dr, dz).$$

Remark 4.14. If μ is a finite-measure and \mathbf{P} -a.s.

$$\int_X \int_{[0,T]} |g_t(x)|^2 dN_t \mu(dx) + \int_X \int_{[0,T]} \int_{U_1} |h_t(x, z)|^2 \eta^p(dt, dz) \mu(dx) < \infty,$$

then (4.6) holds by Hölder's inequality.

4.5 Itô-Wentzell formula

Definition 4.15. We say that an \mathbf{R}^{d_1} -valued \mathbf{F} -adapted quasi-left continuous semimartingale $L_t = (L_t^k)_{1 \leq k \leq d_1}$, $t \geq 0$, is of α -order for some $\alpha \in (0, 2]$ if \mathbf{P} -a.s. for all $t \geq 0$,

$$\sum_{s \leq t} |\Delta L_s|^\alpha < \infty$$

and

$$\begin{aligned} L_t &= L_0 + \int_{[0,t]} \int_{\mathbf{R}_0^{d_1}} z p^L(ds, dz), \text{ if } \alpha \in (0, 1), \\ L_t &= L_0 + A_t + \int_{[0,t]} \int_{|z| \leq 1} z q^L(ds, dz) + \int_{[0,t]} \int_{|z| > 1} z p^L(ds, dz), \text{ if } \alpha \in [1, 2), \\ L_t &= L_0 + A_t + L_t^c + \int_{[0,t]} \int_{|z| \leq 1} z q^L(ds, dz) + \int_{[0,t]} \int_{|z| > 1} z p^L(ds, dz), \text{ if } \alpha = 2, \end{aligned}$$

where $p^L(dt, dz)$ is the jump measure of L with dual predictable projection $\pi^L(dt, dz)$, $q^L(dt, dz) = p^L(dt, dz) - \pi^L(dt, dz)$ is a martingale measure, $A_t = (A_t^i)_{1 \leq i \leq d_1}$ is a continuous process of finite variation with $A_0 = 0$, and $L_t^c = (L_t^{c,i})_{1 \leq i \leq d_1}$ is a continuous local martingale issuing from zero.

Set $(w_s^e)_{s \geq 1} = (w_s^{1:e})_{s \geq 1}$, $(Z, \mathcal{Z}, \pi) = (\mathcal{Z}^1, \mathcal{Z}^1, \pi^1)$, $p(dt, dz) = p^1(dt, dz)$, and $q(dt, dz) = q^1(dt, dz)$. Also, set $D = D^1$, $E = E^1$, and assume $Z = D \cup E$.

Let $f : \Omega \times [0, T] \times \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_2}$ be $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable, $g : \Omega \times [0, T] \times \mathbf{R}^{d_1} \rightarrow \ell_2(\mathbf{R}^{d_2})$ be $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^{d_1}) / \mathcal{B}(\ell_2(\mathbf{R}^{d_2}))$ -measurable, and $h : \Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z \rightarrow \mathbf{R}^{d_2}$ be $\mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}$ -measurable. Moreover, assume that, \mathbf{P} -a.s. for all $x \in \mathbf{R}^{d_1}$,

$$\int_{[0,T]} |f_t(x)| dt + \int_{[0,T]} |g_t(x)|^2 dt + \int_{[0,T]} \int_D |h_t(x, z)|^2 \pi(dz) dt + \int_{[0,T]} \int_E |h_t(x, z)| \pi(dz) dt < \infty.$$

Let $F = F_t(x) : \Omega \times [0, T] \times \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_2}$ be $\mathcal{O}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})$ -measurable and assume that for each x , \mathbf{P} -a.s. for all t ,

$$F_t(x) = F_0(x) + \int_{[0,t]} f_s(x) ds + \int_{[0,t]} g_s^e(x) dw_s^e + \int_{[0,t]} \int_Z h_s(x, z) [\mathbf{1}_D(z) q(ds, dz) + \mathbf{1}_E(z) p(ds, dz)].$$

For each $n \in \{1, 2\}$, let $C_{loc}^n(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ be space of n -times continuously differentiable functions $f : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_2}$. We now state our version of the Itô-Wentzell formula. For each ω, t and x , we denote $\Delta F(x) = F_t(x) - F_{t-}(x)$.

Proposition 4.16 (cf. Proposition 1 in [Mik83]). *Let $(L_t)_{t \geq 0}$ be an \mathbf{R}^{d_1} -valued quasi-left continuous semimartingale of order $\alpha \in (0, 2]$. Assume that:*

- (1) (a) **P**-a.s. $F \in D([0, T]; C_{loc}^\alpha(\mathbf{R}^d; \mathbf{R}^m))$ if α is fractional and $F \in D([0, T]; C_{loc}^\alpha(\mathbf{R}^d; \mathbf{R}^m))$ if $\alpha = 1, 2$;
 (b) for $d\mathbf{P}dt$ -almost-all $(\omega, t) \in \Omega \times [0, T]$, $f_t(x)$ and $g_t(x) = (g_t^{i\varrho}(x))_{\varrho \geq 1} \in \ell_2(\mathbf{R}^{d_2})$ are continuous in x and

$$d\mathbf{P}dt - \lim_{y \rightarrow x} \left[\int_D |h_t(y, z) - h_t(x, z)|^2 \pi(dz) + \int_E |h_t(y, z) - h_t(x, z)| \pi(dz) \right] = 0;$$

- (c) for each $\rho \geq 1$ and $i \in \{1, \dots, d_1\}$ and for $d\mathbf{P}d\langle L^{c:i}, w^\varrho \rangle_t$ -almost-all $(\omega, t) \in \Omega \times [0, T]$, $g_t^{i\varrho} \in C_{loc}^1(\mathbf{R}^d; \mathbf{R})$, if $\alpha = 2$;

- (2) for each compact subset K of \mathbf{R}^{d_1} , **P**-a.s.

$$\begin{aligned} & \int_{[0, T]} \sup_{x \in K} \left(|f_t(x)| + |g_t(x)|^2 + \int_D |h_t(x, z)|^2 \pi(dz) + \int_E |h_t(x, z)| \pi(dz) \right) dt < \infty, \\ & \sum_{\varrho \geq 1} \int_{[0, T]} \sup_{x \in K} |\nabla g_t^{i\varrho}(x)| d\langle L^{c:i}, w^\varrho \rangle_t + \sum_{t \leq T} |\Delta F_t|_{\alpha \wedge 1; K} |\Delta L_t|^{\alpha \wedge 1} < \infty. \end{aligned}$$

Then **P**-a.s for all $t \in [0, T]$,

$$\begin{aligned} F_t(L_t) &= F_0(L_0) + \int_{[0, t]} f_s(L_s) ds + \int_{[0, t]} g_s^\varrho(L_s) dw_s^\varrho \\ &+ \int_{[0, t]} \int_Z h_s(L_{s-}, z) [\mathbf{1}_D(z) q(dr, dz) + \mathbf{1}_E(z) p(dr, dz)] \\ &+ \int_{[0, t]} \partial_i F_{s-}(L_{s-}) [\mathbf{1}_{[1, 2]}(\alpha) dA_s^i + \mathbf{1}_{\{2\}}(\alpha) dL_s^{c:i}] + \mathbf{1}_{\{2\}}(\alpha) \frac{1}{2} \int_{[0, t]} \partial_{ij} F_s(L_s) d\langle L^{c:i}, L^{c:j} \rangle_s \\ &+ \sum_{s \leq t} (F_{s-}(L_s) - F_{s-}(L_{s-}) - \mathbf{1}_{[1, 2]}(\alpha) \nabla F_{s-}(L_{s-}) \Delta L_s) \\ &+ \mathbf{1}_{\{2\}}(\alpha) \int_{[0, t]} \partial_i g_s^\varrho(L_s) d\langle w^\varrho, L^{c:i} \rangle_s + \sum_{s \leq t} (\Delta F_s(L_s) - \Delta F_s(L_{s-})). \end{aligned} \quad (4.7)$$

Proof. Since both sides have identical jumps and we can always interlace a finite set of jumps, we may assume that $|\Delta L_t| \leq 1$ for all $t \in [0, T]$; that is, it is enough to prove the statement for $\tilde{L}_t = L_t - \sum_{s \leq t} \mathbf{1}_{[1, \infty)}(|\Delta L_s|) \Delta L_s$, $t \in [0, T]$. It suffices to assume that for some K and all ω , $|L_0| \leq K$. For each $R > K$, let

$$\tau_R = \inf \left(t \in [0, T] : |A|_t + |\langle L^c \rangle|_t + \sum_{s \leq t} |\Delta L_s|^\alpha + |L_t| > R \right) \wedge T$$

and note that **P**-a.s. $\tau_R \uparrow T$ as R tends to infinity. If instead of L, f, g, h , and F , we take $L_{\cdot \wedge \tau_R}, f \mathbf{1}_{(0, \tau_R]}, g^\varrho \mathbf{1}_{(0, \tau_R]}, h \mathbf{1}_{(0, \tau_R]}, F \mathbf{1}_{(0, \tau_R]}$, then the assumptions of the proposition hold for this

new set of processes. Moreover, if we can prove (4.7) for this new set of processes, then by taking the limit as R tends to infinity, we obtain (4.7). Therefore, we may assume that for some $R > 0$, \mathbf{P} -a.s. for all $t \in [0, T]$,

$$|A|_t + |\langle L^c \rangle|_t + \sum_{s \leq t} |\Delta L_s|^\alpha + |L_t| \leq R. \quad (4.8)$$

Let $\phi \in C_c^\infty(\mathbf{R}^{d_1}, \mathbf{R})$ have support in the unit ball in \mathbf{R}^{d_1} and satisfy $\int_{\mathbf{R}^{d_1}} \phi(x) dx = 1$, $\phi(x) = \phi(-x)$, and $\phi(x) \geq 0$, for all $x \in \mathbf{R}^{d_1}$. For each $\varepsilon \in (0, 1)$, let $\phi_\varepsilon(x) = \varepsilon^{-d} \phi(x/\varepsilon)$, $x \in \mathbf{R}^{d_1}$. By Itô's formula, for each $x \in \mathbf{R}^{d_1}$, \mathbf{P} -a.s. for all $t \in [0, T]$,

$$\begin{aligned} F_t(x) \phi_\varepsilon(x - L_t) &= F_0(x) \phi_\varepsilon(x - L_0) - \int_{]0, t]} F_{s-}(x) \partial_i \phi_\varepsilon(x - L_{s-}) dL_s^i \\ &\quad + \mathbf{1}_{\{2\}}(\alpha) \frac{1}{2} \int_{]0, t]} F_s(x) \partial_{ij} \phi_\varepsilon(x - L_s) d\langle L^{c;i}, L^{c;j} \rangle_s + \int_{]0, t]} \phi_\varepsilon(x - L_s) f_s(x) ds \\ &\quad + \mathbf{1}_{\{2\}}(\alpha) \int_{]0, t]} g_s^\varrho(x) \partial_i \phi_\varepsilon(x - L_s) d\langle w^\varrho, L^{c;i} \rangle_s + \int_{]0, t]} \phi_\varepsilon(x - L_s) g_s^\varrho(x) dw_s^\varrho \\ &\quad + \int_{]0, t]} \int_Z \phi_\varepsilon(x - L_{s-}) h_s(x, z) [\mathbf{1}_D(z) q(dr, dz) + \mathbf{1}_E(z) p(dr, dz)] \\ &\quad + \sum_{s \leq t} \Delta F_s(x) (\phi_\varepsilon(x - L_s) - \phi_\varepsilon(x - L_{s-})) \\ &\quad + \sum_{s \leq t} F_{s-}(x) (\phi_\varepsilon(x - L_s) - \phi_\varepsilon(x - L_{s-}) + \partial_i \phi_\varepsilon(x - L_{s-}) \Delta L_s). \end{aligned}$$

Appealing to assumption (2) and (4.8) (i.e. for the integrals against F), we integrate both sides of the above in x , apply Corollary 4.13 (see, also, Remark 4.14) and the deterministic Fubini theorem, and then integrate by parts to get that \mathbf{P} -a.s. for all $t \in [0, T]$,

$$\begin{aligned} F_t^{(\varepsilon)}(L_t) &= F_0^{(\varepsilon)}(L_0) + \int_{]0, t]} \nabla F_{s-}^{(\varepsilon)}(L_{s-}) [\mathbf{1}_{[1,2]}(\alpha) dA_s^i + \mathbf{1}_{\{2\}}(\alpha) dL_s^{c;i}] + \int_{]0, t]} f_s^{(\varepsilon)}(L_s) dr \\ &\quad + \int_{]0, t]} g_s^{(\varepsilon)}(L_s) dw_s^\varrho + \int_{]0, t]} \int_Z h_s^{(\varepsilon)}(L_{s-}, z) [\mathbf{1}_D(z) q(dr, dz) + \mathbf{1}_E(z) p(dr, dz)] \\ &\quad + \mathbf{1}_{\{2\}}(\alpha) \frac{1}{2} \int_{]0, t]} \partial_{ij} F_s^{(\varepsilon)}(L_s) d\langle L^{c;i}, L^{c;j} \rangle_s + \mathbf{1}_{\{2\}}(\alpha) \int_{]0, t]} \partial_i g_s^{(\varepsilon);\varrho}(L_s) d\langle w^\varrho, L^{c;i} \rangle_s \\ &\quad + \sum_{s \leq t} (\Delta F_s^{(\varepsilon)}(L_s) - \Delta F_s^{(\varepsilon)}(L_{s-})) \\ &\quad + \sum_{s \leq t} (F_{s-}^{(\varepsilon)}(L_s) - F_{s-}^{(\varepsilon)}(L_{s-}) - \mathbf{1}_{[1,2]}(\alpha) \nabla F_{s-}^{(\varepsilon)}(L_{s-}) \Delta L_s) \end{aligned} \quad (4.9)$$

where for each ω, t, x , and z ,

$$F_t^{(\varepsilon)}(x) := \phi_\varepsilon * F_t(x), \quad f_t^{(\varepsilon)} = \phi_\varepsilon * f_t(x), \quad g_t^{(\varepsilon);\varrho}(x) = \phi_\varepsilon * g_t^\varrho(x), \quad h_t^{(\varepsilon)}(x, z) = \phi_\varepsilon * h_t(x, z),$$

and $*$ denotes the convolution operator on \mathbf{R}^{d_1} . Let $B_{R+1} = \{x \in \mathbf{R}^{d_1} : |x| \leq R+1\}$. Owing to assumption (1)(a) and standard properties of mollifiers, for each multi-index γ with $|\gamma| \leq \alpha$, \mathbf{P} -a.s. for all t ,

$$|\partial^\gamma F_t^{(\varepsilon)}(L_t)| \leq \sup_{t \leq T} \sup_{x \in B_{R+1}} |\partial^\gamma F_t(x)| < \infty$$

and for each x ,

$$d\mathbf{P}dt - \lim_{\varepsilon \downarrow 0} |\partial^\gamma F_t^{(\varepsilon)}(x) - \partial^\gamma F_t(x)| = 0.$$

Similarly, by assumption 1(b), $d\mathbf{P}dt$ -almost-all $(\omega, t) \in \Omega \times [0, T]$,

$$\begin{aligned} |f_t^{(\varepsilon)}(L_t)| &\leq \sup_{x \in B_{R+1}} |f_t(x)| < \infty, \quad |g_t^{(\varepsilon)}(L_t)| \leq \sup_{x \in B_{R+1}} |g_t(x)| < \infty, \\ \int_D |h_t^\varepsilon(L_t, z)|^2 \pi(dz) &\leq \sup_{x \in B_{R+1}} \int_D |h_t(x, z)|^2 \pi(dz), \\ \int_E |h_t^\varepsilon(L_t, z)| \pi(dz) &\leq \sup_{x \in B_{R+1}} \int_E |h_t(x, z)| \pi(dz) \end{aligned}$$

and for each x ,

$$d\mathbf{P}dt - \lim_{\varepsilon \downarrow 0} |f_t^{(\varepsilon)}(x) - f_t(x)| = 0, \quad d\mathbf{P}dt - \lim_{\varepsilon \rightarrow 0} |g_t^{(\varepsilon)}(x) - g_t(x)| = 0$$

and

$$d\mathbf{P}dt - \lim_{\varepsilon \downarrow 0} \int_Z [\mathbf{1}_D(z) |h_t^{(\varepsilon)}(x, z) - h_t(x, z)|^2 + \mathbf{1}_E(z) |h_t^{(\varepsilon)}(x, z) - h_t(x, z)|] \pi(dz) = 0,$$

where in the last-line we have also used Minkowski's integral inequality and a standard mollifying convergence argument. Using assumption 1(d), for each $\rho \geq 1$ and $i \in \{1, \dots, d_1\}$ and for $d\mathbf{P}d\langle L^{c;i}, w^\rho \rangle_t$ -almost-all $(\omega, t) \in \Omega \times [0, T]$

$$|\nabla g_t^{(\varepsilon); i\rho}(L_t)| \leq \sup_{x \in B_{R+1}} |\nabla g_t^{i\rho}(x)|$$

and for each x ,

$$d\mathbf{P}d\langle L^{c;i}, w^\rho \rangle_t - \lim_{\varepsilon \rightarrow 0} |\nabla g_t^{(\varepsilon); i\rho}(x) - \nabla g_t^{i\rho}(x)| = 0, \quad \text{if } \alpha = 2.$$

Owing to assumption 1(a) and (4.8), \mathbf{P} -a.s.

$$\begin{aligned} &\sum_{s \leq t} |F_{s-}^{(\varepsilon)}(L_s) - F_{s-}^{(\varepsilon)}(L_{s-}) - \mathbf{1}_{[1,2]}(\alpha) \nabla F_{s-}^{(\varepsilon)}(L_{s-}) \Delta L_s| \\ &\leq \sup_{t \leq T} |F_t|_{\alpha; B_{R+1}} \sum_{s \leq t} |\Delta L_s|^\alpha \leq R \sup_{t \leq T} |F_t|_{\alpha; B_{R+1}}. \end{aligned}$$

Since \mathbf{P} -a.s. $F \in D([0, T]; C^\alpha(\mathbf{R}^d; \mathbf{R}^m))$, it follows that for each x , \mathbf{P} -a.s. for all t ,

$$\lim_{\varepsilon \downarrow 0} |\Delta F_t^\varepsilon(x) - \Delta F_t(x)| = 0.$$

By assumption (2), \mathbf{P} -a.s for all t , we have

$$\sum_{s \leq t} \left(\Delta F_s^{(\varepsilon)}(L_{s-} + \Delta L_s) - \Delta F_s^{(\varepsilon)}(L_{s-}) \right) \leq \sum_{s \leq t} |\Delta F_t|_{\alpha \wedge 1; B_{R+1}} |\Delta L_s|^{\alpha \wedge 1}.$$

Combining the above and using assumptions (1)(a) and (2) and the bounds given in (4.8) and the deterministic and stochastic dominated convergence theorem, we obtain convergence of all the terms in (4.9), which complete the proof. \square

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